

DOUBLE LIE ALGEBROIDS AND REPRESENTATIONS UP TO HOMOTOPY

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ABSTRACT. We show that double Lie algebroids, together with a chosen linear splitting, are equivalent to pairs of 2-term representations up to homotopy satisfying compatibility conditions which extend the notion of matched pair of Lie algebroids. We discuss in detail the tangent of a Lie algebroid.

CONTENTS

1. Introduction	1
2. Background and definitions	3
2.1. Double vector bundles, decompositions and dualization	3
2.2. VB-algebroids and double Lie algebroids	6
2.3. Representations up to homotopy and VB-algebroids	7
3. Main theorem and examples	9
3.1. Matched pairs of representations up to homotopy and main result	10
3.2. The tangent double of a Lie algebroid	11
4. Proof of the theorem	12
4.1. Families of sections of Lie bialgebroids	13
4.2. The Lie bialgebroid conditions on lifts and on core sections	14
References	20

1. INTRODUCTION

Double Lie algebroids first arose as the infinitesimal form of double Lie groupoids [9, 12]. In the same way as the Lie theory of Lie groupoids and Lie algebroids expresses many of the basic infinitesimalization and integration results of differential geometry, the process of taking the double Lie algebroid of a double Lie groupoid captures two-stage differentiation processes, such as the iterated tangent bundle of a smooth manifold, and the relations between a Poisson Lie group, its Lie bialgebra and its symplectic double groupoid.

The transition from a double Lie groupoid to its double Lie algebroid is straightforward. To define an abstract concept of double Lie algebroid, however, is much more difficult, since there is no meaningful way in which a Lie algebroid bracket can be said to be a morphism with respect to another Lie algebroid structure. The solution ultimately found was to extend the duality between Lie algebroids and Lie-Poisson structures to the double context, using the duality properties of double vector bundles [14]. This definition was immediately given a simple and elegant reformulation in terms of super geometry and coordinates by Th. Voronov [20]. In terms of super geometry, a Lie algebroid structure on a vector bundle corresponds

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to a homological vector field Q of weight 1 on the parity-reversed bundle. A double vector bundle D with Lie algebroid structures on both bundle structures on D therefore involves two homological vector fields, of suitable weights, and the main compatibility condition of [20] is that they commute.

In the present paper we give a third formulation of double Lie algebroids, in terms of representations up to homotopy as defined in [5] and [1]; this differs from the concept of [3]. In fact the representations up to homotopy which are relevant are concentrated in degrees 0 and 1 and we refer to them as *2-representations* for brevity. Consider first a double vector bundle D with Lie algebroid structures on two parallel sides, which are compatible with the vector bundle structures on the other sides; these are variously called \mathcal{LA} -groupoids or VB-groupoids, and are the ‘preliminary case’ of double Lie algebroids in [14]. Further suppose that D is ‘decomposed’; that is, as a manifold it is the fibre product $A \times_M B \times_M C$ of three vector bundles A, B, C on the same base M , and the vector bundle structures on D are the pullbacks of $B \oplus C$ to A and of $A \oplus C$ to B . Then [5] showed that VB-groupoid structures on D are in bijective correspondence with 2-representations defined in terms of A, B and C .

It is always possible to ‘decompose’ a double vector bundle; that is, any double vector bundle is isomorphic to a decomposed double vector bundle. Decompositions may be regarded as trivializations of D at the double level; in this paper we do not need to trivialize A, B and C . For a formulation in coordinate terms, see [20].

Now consider an arbitrary double Lie algebroid D . The Lie algebroid structures on D may be considered as a pair of VB-structures and accordingly a decomposition of D expresses the double Lie algebroid structure as a pair of 2-representations. Our main result (Theorem 3.4) determines the compatibility conditions between these and, conversely, proves that a suitable pair of 2-representations defines a double Lie algebroid structure on D .

Our formulation is in some respects midway between the original formulation and that of [20]. Our treatment resembles that of Voronov inasmuch as the three intrinsic conditions of [14] are replaced by a greater number of conditions which are dependent on auxiliary data, but are easier to work with. On the other hand, our methods are entirely ‘classical’ rather than supergeometric, and rely on a global decomposition rather than local coordinates.

Our formulation may also be regarded as a considerable generalization of the description of a vacant double Lie algebroid in terms of a matched pair of representations [14, §6] — that is, of representations of Lie algebroids in the strict sense, without curvature. For this reason we regard the conditions (M1) to (M9) in Definition 3.1 as defining a matched pair of 2-representations.

In turn, [6] will show that the bicrossproduct of a matched pair of 2-representations is a split Lie 2-algebroid, in the same way that the bicrossproduct of a matched pair of representations of Lie algebroids is a Lie algebroid [16, 8]. In a different direction, [7] will apply our main result to show that double Lie algebroids which are transitive in a sense appropriate to the double structure are determined by a simple diagram of morphisms of ordinary Lie algebroids.

We now describe the contents of the paper.

In §2 we recall the basic notions needed throughout the paper. We begin with double vector bundles, the special classes of sections with which it is easiest to work, and the nonstandard pairing between their duals. In §2.2 we recall VB-algebroids and double Lie algebroids, and in §2.3 we finally define 2-representations.

In §3 we state our main result and the main work of the proof is given in §4.

We have included definitions of the key concepts required; in particular it is not necessary to have a detailed knowledge of [1], [5], [14] or [20].

2. BACKGROUND AND DEFINITIONS

2.1. Double vector bundles, decompositions and dualization. We briefly recall the definitions of double vector bundles, of their *linear* and *core* sections, and of their *linear splittings* and *lifts*. We refer to [18, 13, 5] for more detailed treatments.

Definition 2.1. A *double vector bundle* is a commutative square

$$\begin{array}{ccc} D & \xrightarrow{\pi_B} & B \\ \pi_A \downarrow & & \downarrow q_B \\ A & \xrightarrow{q_A} & M \end{array}$$

satisfying the following four conditions:

- (1) all four sides are vector bundles;
- (2) π_B is a vector bundle morphism over q_A ;
- (3) $+_B : D \times_B D \rightarrow D$ is a vector bundle morphism over $+ : A \times_M A \rightarrow A$, where $+_B$ is the addition map for the vector bundle $D \rightarrow B$, and
- (4) the scalar multiplication $\mathbb{R} \times D \rightarrow D$ in the bundle $D \rightarrow B$ is a vector bundle morphism over the scalar multiplication $\mathbb{R} \times A \rightarrow A$.

The corresponding statements for the operations in the bundle $D \rightarrow A$ follow.

Given a double vector bundle $(D; A, B; M)$, the vector bundles A and B are called the **side bundles**. The **core** C of a double vector bundle is the intersection of the kernels of π_A and π_B . It has a natural vector bundle structure over M , the restriction of either structure on D , the projection of which we call $q_C : C \rightarrow M$. The inclusion $C \hookrightarrow D$ is usually denoted by

$$C_m \ni c \mapsto \bar{c} \in \pi_A^{-1}(0_m^A) \cap \pi_B^{-1}(0_m^B).$$

Given a double vector bundle $(D; A, B; M)$, the space of sections $\Gamma_B(D)$ is generated as a $C^\infty(B)$ -module by two distinguished classes of sections (see [14]), the *linear* and the *core sections* which we now describe.

Definition 2.2. For a section $c : M \rightarrow C$, the corresponding **core section** $c^\dagger : B \rightarrow D$ is defined as

$$(1) \quad c^\dagger(b_m) = \tilde{0}_{b_m} +_A \overline{c(m)}, \quad m \in M, b_m \in B_m.$$

We denote the corresponding core section $A \rightarrow D$ by c^\dagger also, relying on the argument to distinguish between them.

Definition 2.3. A section $\xi \in \Gamma_B(D)$ is called **linear** if $\xi : B \rightarrow D$ is a bundle morphism from $B \rightarrow M$ to $D \rightarrow A$ over a section $a \in \Gamma(A)$.

The space of core sections of D over B will be written $\Gamma_B^c(D)$ and the space of linear sections $\Gamma_B^\ell(D)$. Given $\psi \in \Gamma(B^* \otimes C)$, there is a linear section $\tilde{\psi} : B \rightarrow D$ over the zero section $0^A : M \rightarrow A$ given by

$$(2) \quad \tilde{\psi}(b_m) = \tilde{0}_{b_m} +_A \overline{\psi(b_m)}.$$

We call $\tilde{\psi}$ a **core-linear section**.

Example 2.4. Let A, B, C be vector bundles over M and consider $D = A \times_M B \times_M C$. With the vector bundle structures $D = q_A^!(B \oplus C) \rightarrow A$ and $D = q_B^!(A \oplus C) \rightarrow B$, one finds that $(D; A, B; M)$ is a double vector bundle called the *decomposed* or *trivial double vector bundle with core C*. The core sections are given by

$$c^\dagger : b_m \mapsto (0_m^A, b_m, c(m)), \quad \text{where } m \in M, b_m \in B_m, c \in \Gamma(C),$$

and similarly for $c^\dagger: A \rightarrow D$. The space of linear sections $\Gamma_B^\ell(D)$ is naturally identified with $\Gamma(A) \oplus \Gamma(B^* \otimes C)$ via

$$(a, \psi) : b_m \mapsto (a(m), b_m, \psi(b_m)), \text{ where } \psi \in \Gamma(B^* \otimes C), a \in \Gamma(A).$$

In particular, the fibered product $A \times_M B$ is a double vector bundle over the sides A and B and has core $M \times 0$.

2.1.1. Linear splittings and lifts. A **linear splitting**¹ of $(D; A, B; M)$ is an injective morphism of double vector bundles $\Sigma: A \times_M B \hookrightarrow D$ over the identity on the sides A and B . That every double vector bundle admits local linear splittings was proved by [4]. Local linear splittings are equivalent to double vector bundle charts. Pradines originally defined double vector bundles as topological spaces with an atlas of double vector bundle charts [17]. Using a partition of unity, he proved that (provided the double base is a smooth manifold) this implies the existence of a global double splitting [18]. Hence, any double vector bundle in the sense of Definition 2.1 admits a (global) linear splitting.

A linear splitting Σ of D is equivalent to a splitting σ_A of the short exact sequence of $C^\infty(M)$ -modules

$$0 \longrightarrow \Gamma(B^* \otimes C) \hookrightarrow \Gamma_B^\ell(D) \longrightarrow \Gamma(A) \longrightarrow 0,$$

where the third map is the map that sends a linear section (ξ, a) to its base section $a \in \Gamma(A)$. The splitting σ_A will be called a **lift**. Given Σ , the lift $\sigma_A: \Gamma(A) \rightarrow \Gamma_B^\ell(D)$ is given by $\sigma_A(a)(b_m) = \Sigma(a(m), b_m)$ for all $a \in \Gamma(A)$ and $b_m \in B$.

In the case of the tangent double of a vector bundle $E \rightarrow M$, the lift from vector fields on M to vector fields on E (see 2.1.2) would be the horizontal lift corresponding to a connection. We avoid the word ‘horizontal’ here since ‘horizontal’ and ‘vertical’ refer to the two structures on D .

By the symmetry of a linear splitting, this implies that a lift $\sigma_A: \Gamma(A) \rightarrow \Gamma_B^\ell(D)$ is equivalent to a lift $\sigma_B: \Gamma(B) \rightarrow \Gamma_A^\ell(D)$. Given a lift $\sigma_A: \Gamma(A) \rightarrow \Gamma_B^\ell(D)$, the corresponding lift $\sigma_B: \Gamma(B) \rightarrow \Gamma_A^\ell(D)$ is given by $\sigma_B(b)(a(m)) = \sigma_A(a)(b(m))$ for all $a \in \Gamma(A)$, $b \in \Gamma(B)$.

Note finally that two linear splittings $\Sigma^1, \Sigma^2: A \times_M B \rightarrow D$ differ by a section Φ_{12} of $A^* \otimes B^* \otimes C \simeq \text{Hom}(A, B^* \otimes C) \simeq \text{Hom}(B, A^* \otimes C)$ in the following sense. For each $a \in \Gamma(A)$ the difference $\sigma_A^1(a) - \sigma_A^2(a)$ of lifts is the core-linear section defined by $\Phi_{12}(a) \in \Gamma(B^* \otimes C)$. By symmetry, $\sigma_B^1(b) - \sigma_B^2(b) = \widetilde{\Phi_{12}(b)}$ for each $b \in \Gamma(B)$.

2.1.2. The tangent double of a vector bundle. Let $q_E: E \rightarrow M$ be a vector bundle. Then the tangent bundle TE has two vector bundle structures; one as the tangent bundle of the manifold E , and the second as a vector bundle over TM . The structure maps of $TE \rightarrow TM$ are the derivatives of the structure maps of $E \rightarrow M$. The space TE is a double vector bundle with core bundle $E \rightarrow M$.

$$\begin{array}{ccc} TE & \xrightarrow{p_E} & E \\ Tq_E \downarrow & & \downarrow q_E \\ TM & \xrightarrow{p_M} & M \end{array}$$

The core vector field corresponding to $e \in \Gamma(E)$ is the vertical lift $e^\uparrow: E \rightarrow TE$, i.e. the vector field with flow $\phi: E \times \mathbb{R} \rightarrow E$, $\phi_t(e'_m) = e'_m + te(m)$. An element of $\Gamma_E^\ell(TE) = \mathfrak{X}^\ell(E)$ is called a **linear vector field**. It is well-known (see e.g. [13]) that a linear vector field $\xi \in \mathfrak{X}^\ell(E)$ covering $X \in \mathfrak{X}(M)$ corresponds to a derivation $D^*: \Gamma(E^*) \rightarrow \Gamma(E^*)$ over $X \in$

¹Note that a linear splitting of D is equivalent to a **decomposition** of D , i.e. an isomorphism $\mathbb{I}: A \times_M B \times_M C \rightarrow D$ of double vector bundles over the identities on the sides and core. Given a linear splitting Σ , the corresponding decomposition \mathbb{I} is given by $\mathbb{I}(a_m, b_m, c_m) = \Sigma(a_m, b_m) +_B (\bar{0}_{b_m} +_A \bar{c}_m)$. Given a decomposition \mathbb{I} , the corresponding linear splitting Σ is given by $\Sigma(a_m, b_m) = \mathbb{I}(a_m, b_m, 0_m^C)$.

$\mathfrak{X}(M)$, and hence to a derivation $D: \Gamma(E) \rightarrow \Gamma(E)$ over $X \in \mathfrak{X}(M)$ (the dual derivation). The precise correspondence is given by²

$$(3) \quad \xi(\ell_\varepsilon) = \ell_{D^*(\varepsilon)} \quad \text{and} \quad \xi(q_E^* f) = q_E^*(X(f))$$

for all $\varepsilon \in \Gamma(E^*)$ and $f \in C^\infty(M)$. Here ℓ_ε is the linear function $E \rightarrow \mathbb{R}$ corresponding to ε . We will write \widehat{D} for the linear vector field corresponding in this manner to a derivation D of $\Gamma(E)$. The choice of a linear splitting Σ for $(TE; TM, E; M)$ is equivalent to the choice of a connection on E : Since a linear splitting gives us a linear vector field $\sigma_{TM}(X) \in \mathfrak{X}^l(E)$ for each X , we can define $\nabla: \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ by $\sigma_{TM}(X) = \widehat{\nabla_X}$ for all $X \in \mathfrak{X}(M)$. Conversely, a connection $\nabla: \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ defines a lift σ_{TM}^∇ for $(TE; TM, E; M)$ and a linear splitting $\Sigma^\nabla: TM \times_M E \rightarrow TE$.

We recall as well the relation between the connection and the Lie bracket of vector fields on E . Given ∇ , it is easy to see using the equalities in (3) that, writing σ for σ_{TM}^∇ :

$$(4) \quad [\sigma(X), \sigma(Y)] = \sigma[X, Y] - R_\nabla(X, Y)^\uparrow, \quad [\sigma(X), e^\uparrow] = (\nabla_X e)^\uparrow, \quad [e^\uparrow, e'^\uparrow] = 0,$$

for all $X, Y \in \mathfrak{X}(M)$ and $e, e' \in \Gamma(E)$. That is, the Lie bracket of vector fields on M and the connection encode completely the Lie bracket of vector fields on E .

Now let us have a quick look at the other structure on the double vector bundle TE . The lift $\sigma_E^\nabla: \Gamma(E) \rightarrow \Gamma_{TM}^\ell(TE)$ is given by

$$(5) \quad \sigma_E^\nabla(e)(v) = Te(v) +_{TM} (T0^E(v) -_E \overline{\nabla_v e}), \quad v \in TM, e \in \Gamma(E).$$

2.1.3. Dualization and lifts. Recall that double vector bundles can be dualized in two distinct ways. We denote the dual of D as a vector bundle over A by $D \star A$ and likewise for $D \star B$. The dual $D \star A$ is itself a double vector bundle, with side bundles A and C^* and core B^* [11, 14].

$$\begin{array}{ccc} \begin{array}{ccc} D & \xrightarrow{\pi_B} & B \\ \pi_A \downarrow & & \downarrow q_B \\ A & \xrightarrow{q_A} & M \end{array} & \begin{array}{ccc} D \star A & \xrightarrow{\pi_{C^*}} & C^* \\ \pi_A \downarrow & & \downarrow q_{C^*} \\ A & \xrightarrow{q_A} & M \end{array} & \begin{array}{ccc} D \star B & \xrightarrow{\pi_B} & B \\ \pi_{C^*} \downarrow & & \downarrow q_B \\ C^* & \xrightarrow{q_{C^*}} & M \end{array} \end{array}$$

Note also that by dualizing again $D \star B$ over C^* , we get

$$\begin{array}{ccc} D \star B \star C^* & \xrightarrow{\pi_A} & A \\ \pi_{C^*} \downarrow & & \downarrow q_A \\ C^* & \xrightarrow{q_{C^*}} & M, \end{array}$$

with core B^* . In the same manner, we get a double vector bundle $D \star A \star C^*$ with sides B and C^* and core A^* .

Recall first of all that the vector bundles $D \star B \rightarrow C^*$ and $D \star A \rightarrow C^*$ are, up to a sign, naturally in duality to each other [13]. The pairing

$$\langle \cdot, \cdot \rangle: (D \star A) \times_{C^*} (D \star B) \rightarrow \mathbb{R}$$

is defined as follows: for $\Phi \in D \star A$ and $\Psi \in D \star B$ projecting to the same element γ_m in C^* , choose $d \in D$ with $\pi_A(d) = \pi_A(\Phi)$ and $\pi_B(d) = \pi_B(\Psi)$. Then $\langle \Phi, d \rangle_A - \langle \Psi, d \rangle_B$ does not depend on the choice of d and we set $\langle \Phi, \Psi \rangle = \langle \Phi, d \rangle_A - \langle \Psi, d \rangle_B$.

This implies in particular that $D \star A$ is canonically (up to a sign) isomorphic to $D \star B \star C^*$ and $D \star B$ is isomorphic to $D \star A \star C^*$. We will use this below.

²Since its flow is a flow of vector bundle morphisms, a linear vector field sends linear functions to linear functions and pullbacks to pullbacks.

Each linear section $\xi \in \Gamma_B(D)$ over $a \in \Gamma(A)$ induces a linear section $\xi^\square \in \Gamma_{C^*}^\ell(D \star B \star C^*)$ over a . Namely ξ induces a function $\ell_\xi: D \star B \rightarrow \mathbb{R}$ which is fibrewise-linear over B and, using the definition of the addition in $D \star B \rightarrow C^*$ ([14, Equation (7)], it follows that ℓ_ξ is also linear over C^* . The corresponding section of $D \star B \star C^* \rightarrow C^*$ is denoted ξ^\square [14]. Thus

$$(6) \quad \langle \xi^\square(\gamma), \Psi \rangle = \ell_\xi(\Psi) = \langle \Psi, \xi(b) \rangle_B$$

for $\Psi \in D \star B$ such that $\pi_B(\Psi) = b$ and $\pi_{C^*}(\Psi) = \gamma$.

Given a linear splitting $\Sigma: A \times_M B \rightarrow D$ of D , we get hence a linear splitting $\Sigma^{*,B}: C^* \times_M A \rightarrow D \star B \star C^*$, defined by the corresponding lift $\sigma_A^{*,B}: \Gamma(A) \rightarrow \Gamma_{C^*}^\ell(D \star B \star C^*)$:

$$\sigma_A^{*,B}(a) = (\sigma_A(a))^\square$$

for all $a \in \Gamma(A)$.

We now use the (canonical up to a sign) isomorphism of $D \star A$ with $D \star B \star C^*$ to construct a canonical linear splitting of $D \star A$ given a linear splitting of D . We identify $D \star A$ with $D \star B \star C^*$ using $-\langle \cdot, \cdot \rangle$. Thus we define the lift $\sigma_A^*: \Gamma(A) \rightarrow \Gamma_{C^*}^\ell(D \star A)$ by

$$(7) \quad \langle \sigma_A^*(a), \cdot \rangle = -\sigma_A^{*,B}(a)$$

for all $a \in \Gamma(A)$. Note that by (6), this implies that $\langle \sigma_A^*(a)(\gamma), \sigma_A(a)(b) \rangle_A = 0$ for all $\gamma \in C^*$ and $b \in B$. The choice of sign in (7) is necessary for $\sigma_A^*(a)$ to be a linear section of $D \star A$ over a . To be more explicit, check or recall from [13, Equation (28), p.352] that $\langle \sigma_A^*(a), \alpha^\dagger \rangle = -\langle \alpha, a \rangle$ for all $\alpha \in \Gamma(A^*)$ (and α^\dagger the corresponding core section of $D \star B$ over C^*). But $\langle \sigma_A^{*,B}(a), \alpha^\dagger \rangle_{C^*} = q_{C^*}^* \langle a, \alpha \rangle$ by definition of the pairing of $D \star B \star C^*$ with $D \star B$. Hence, without the choice of sign that we make, $\sigma_A^*(a)$ would be linear over $-a$, hence *not* a lift.

By (skew-)symmetry, given the lift $\sigma_B: \Gamma(B) \rightarrow \Gamma_B^\ell(D)$, we identify $D \star B$ with $D \star A \star C^*$ using $\langle \cdot, \cdot \rangle$ and define the lift $\sigma_B^*: \Gamma(B) \rightarrow \Gamma_{C^*}^\ell(D \star B)$ by $\langle \sigma_B^*(b), \cdot \rangle = \sigma_B^{*,A}(b)$ for all $b \in \Gamma(B)$. (This time, we do not need the minus sign.) As a summary, we have the equations:

$$(8) \quad \langle \sigma_A^*(a), \sigma_B^*(b) \rangle = 0, \quad \langle \sigma_A^*(a), \alpha^\dagger \rangle = -q_{C^*}^* \langle \alpha, a \rangle, \quad \langle \beta^\dagger, \sigma_B^*(b) \rangle = q_{C^*}^* \langle \beta, b \rangle,$$

for all $a \in \Gamma(A)$, $b \in \Gamma(B)$, $\alpha \in \Gamma(A^*)$ and $\beta \in \Gamma(B^*)$.

2.2. VB-algebroids and double Lie algebroids. What we are here calling VB-algebroids were defined in [10, 14] and called \mathcal{LA} -vector bundles.³

Let $(D; A, B; M)$ be a double vector bundle

$$\begin{array}{ccc} D & \xrightarrow{\pi_B} & B \\ \pi_A \downarrow & & \downarrow q_B \\ A & \xrightarrow{q_A} & M \end{array}$$

with core C .

Then $(D; A, B; M)$ is a VB-algebroid ([10]; see also [5]) if there are Lie algebroid structures on $D \rightarrow B$ and $A \rightarrow M$, such that the vector bundle operations in $D \rightarrow A$ are Lie algebroid morphisms over the corresponding operations in $B \rightarrow M$. The anchor $\Theta_B: D \rightarrow TB$ is a morphism of double vector bundles and we denote the induced map of the cores by $\partial: C \rightarrow B$. Equivalently [5], $(D \rightarrow B; A \rightarrow M)$ is a VB-algebroid if $D \rightarrow B$ is a Lie algebroid, the anchor $\Theta_D: D \rightarrow TB$ is a bundle morphism over $\rho_A: A \rightarrow TM$ and the Lie bracket is linear:

$$[\Gamma_B^\ell(D), \Gamma_B^\ell(D)] \subset \Gamma_B^\ell(D), \quad [\Gamma_B^\ell(D), \Gamma_B^c(D)] \subset \Gamma_B^c(D), \quad [\Gamma_B^c(D), \Gamma_B^c(D)] = 0.$$

³The terminology ‘ \mathcal{LA} -vector bundle’ followed that of \mathcal{LA} -groupoids, which were defined in [9, §4], on the model of Pradines’ [19] concept of \mathcal{VB} -groupoid. In [10] and [14, 3.3] \mathcal{LA} -vector bundles were seen as a special case of \mathcal{LA} -groupoids. The terminology ‘VB-algebroid’ of [5] distinguishes the equivalent formulation in terms of bracket conditions on the linear and core sections.

The vector bundle $A \rightarrow M$ is then also a Lie algebroid, with anchor ρ_A and bracket defined as follows: if $\xi_1, \xi_2 \in \Gamma_B^\ell(D)$ are linear over $a_1, a_2 \in \Gamma(A)$, then the bracket $[\xi_1, \xi_2]$ is linear over $[a_1, a_2]$.

Example 2.5. The tangent double $(TE; E, TM; M)$ has a VB-algebroid structure $(TE \rightarrow E, TM \rightarrow M)$.

If D is a VB-algebroid with Lie algebroid structures on $D \rightarrow B$ and $A \rightarrow M$ the dual vector bundle $D \star B \rightarrow B$ has a *Lie-Poisson structure* (a linear Poisson structure), and the structure on $D \star B$ is also Lie-Poisson with respect to $D \star B \rightarrow C^*$ [14, 3.4]. Dualizing this bundle gives a Lie algebroid structure on $D \star B \star C^* \rightarrow C^*$. This equips the double vector bundle $(D \star B \star C^*; C^*, A; M)$ with a VB-algebroid structure. Using the isomorphism defined by $-\langle \cdot, \cdot \rangle$, the double vector bundle $(D \star A; A, C^*; M)$ also has a VB-algebroid structure.

Definition 2.6 ([14]). A **double Lie algebroid** is a double vector bundle $(D; A, B; M)$ with core denoted C , and with Lie algebroid structures on each of $A \rightarrow M$, $B \rightarrow M$, $D \rightarrow A$ and $D \rightarrow B$ such that each pair of parallel Lie algebroids gives D the structure of a VB-algebroid, and such that $(D \star A \star C^*, D \star B \star C^*)$ with the induced Lie algebroid structures on base C^* as defined above, is a Lie bialgebroid.

Equivalently, D is a double Lie algebroid if the pair $(D \star A, D \star B)$ with the induced Lie algebroid structures on base C^* and the pairing $\langle \cdot, \cdot \rangle$, is a Lie bialgebroid. One aim of this paper is to reformulate this definition in terms of specific classes of sections, so as to allow the user to bypass frequent use of the duality of doubles; see Theorem 3.4.

2.3. Representations up to homotopy and VB-algebroids. Let $A \rightarrow M$ be a Lie algebroid and consider an A -connection ∇ on a vector bundle $E \rightarrow M$. Then the space $\Omega^\bullet(A, E)$ of E -valued Lie algebroid forms has an induced operator \mathbf{d}_∇ given by the Koszul formula:

$$\begin{aligned} \mathbf{d}_\nabla \omega(a_1, \dots, a_{k+1}) &= \sum_{i < j} (-1)^{i+j} \omega([a_i, a_j], a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_{k+1}) \\ &\quad + \sum_i (-1)^{i+1} \nabla_{a_i} (\omega(a_1, \dots, \hat{a}_i, \dots, a_{k+1})) \end{aligned}$$

for all $\omega \in \Omega^k(A, E)$ and $a_1, \dots, a_{k+1} \in \Gamma(A)$.

Let now $\mathcal{E} = \bigoplus_{k \in \mathbb{Z}} E_k[k]$ be a graded vector bundle. Consider the space $\Omega(A, \mathcal{E})$ with grading given by

$$\Omega(A, \mathcal{E})^k = \bigoplus_{i+j=k} \Omega^i(A, E_j).$$

Definition 2.7. [1] A representation up to homotopy of A on \mathcal{E} is a map $\mathcal{D}: \Omega(A, \mathcal{E}) \rightarrow \Omega(A, \mathcal{E})$ with total degree 1 and such that $\mathcal{D}^2 = 0$ and

$$\mathcal{D}(\alpha \wedge \omega) = \mathbf{d}_A \alpha \wedge \omega + (-1)^{|\alpha|} \alpha \wedge \mathcal{D}(\omega), \text{ for } \alpha \in \Gamma(\wedge A^*), \omega \in \Omega(A, \mathcal{E}),$$

where $\mathbf{d}_A: \Gamma(\wedge A^*) \rightarrow \Gamma(\wedge A^*)$ is the Lie algebroid differential.

Note that Gracia-Saz and Mehta [5] defined this concept independently and called them “superrepresentations”.

Let A be a Lie algebroid. The representations up to homotopy which we will consider are always on graded vector bundles $\mathcal{E} = E_0 \oplus E_1$ concentrated on degrees 0 and 1, so called *2-term graded vector bundles*. These representations are equivalent to the following data (see [1, 5]):

- (1) a map $\partial: E_0 \rightarrow E_1$,
- (2) two A -connections, ∇^0 and ∇^1 on E_0 and E_1 , respectively, such that $\partial \circ \nabla^0 = \nabla^1 \circ \partial$,

- (3) an element $R \in \Omega^2(A, \text{Hom}(E_1, E_0))$ such that $R_{\nabla^0} = R \circ \partial$, $R_{\nabla^1} = \partial \circ R$ and $\mathbf{d}_{\nabla^{\text{Hom}}} \omega = 0$, where ∇^{Hom} is the connection induced on $\text{Hom}(E_1, E_0)$ by ∇^0 and ∇^1 .

We will call such a 2-term representation up to homotopy a **2-representation** for brevity.

Let $(D \rightarrow B; A \rightarrow M)$ be a VB-algebroid. Then since the anchor ρ_D is linear, it sends a core section c^\dagger , $c \in \Gamma(C)$ to a vertical vector field on B . This defines the **core-anchor** $\partial_B: C \rightarrow B$ given by, $\rho_D(c^\dagger) = (\partial_B c)^\dagger$ for all $c \in \Gamma(C)$ (see [9]).

Choose further a linear splitting $\Sigma: A \times_M B \rightarrow D$. Since the anchor of a linear section is linear, for each $a \in \Gamma(A)$ the vector field $\rho_D(\sigma_A(a))$ defines a derivation of $\Gamma(B)$ with symbol $\rho(a)$ (see §2.1.2). This defines a linear connection $\nabla^{AB}: \Gamma(A) \times \Gamma(B) \rightarrow \Gamma(B)$:

$$\rho_D(\sigma_A(a)) = \widehat{\nabla_a^{AB}}$$

for all $a \in \Gamma(A)$. Since the bracket of a linear section with a core section is again linear, we find a linear connection $\nabla^{AC}: \Gamma(A) \times \Gamma(C) \rightarrow \Gamma(C)$ such that

$$[\sigma_A(a), c^\dagger] = (\nabla_a^{AC} c)^\dagger$$

for all $c \in \Gamma(C)$ and $a \in \Gamma(A)$. The difference $\sigma_A[a_1, a_2] - [\sigma_A(a_1), \sigma_A(a_2)]$ is a core-linear section for all $a_1, a_2 \in \Gamma(A)$. This defines a vector valued Lie algebroid form $R \in \Omega^2(A, \text{Hom}(B, C))$ such that

$$[\sigma_A(a_1), \sigma_A(a_2)] = \sigma_A[a_1, a_2] - \widetilde{R(a_1, a_2)},$$

for all $a_1, a_2 \in \Gamma(A)$. See [5] for more details on these constructions.

The following theorem is proved in [5].

Theorem 2.8. *Let $(D \rightarrow B; A \rightarrow M)$ be a VB-algebroid and choose a linear splitting $\Sigma: A \times_M B \rightarrow D$. The triple $(\nabla^{AB}, \nabla^{AC}, R)$ defined as above is a 2-representation of A on the complex $\partial_B: C \rightarrow B$.*

Conversely, let $(D; A, B; M)$ be a double vector bundle such that A has a Lie algebroid structure and choose a linear splitting $\Sigma: A \times_M B \rightarrow D$. Then if $(\nabla^{AB}, \nabla^{AC}, R)$ is a 2-representation of A on a complex $\partial_B: C \rightarrow B$, then the four equations above define a VB-algebroid structure on $(D \rightarrow B; A \rightarrow M)$.

The following formulas for the brackets of linear and core sections with core-linear sections will be very useful in the proof of our main theorem. In the situation of the previous theorem, we have

$$(9) \quad [\sigma_A(a), \widetilde{\phi}] = \widetilde{\nabla_a^{\text{Hom}} \phi}$$

and

$$(10) \quad [c^\dagger, \widetilde{\phi}] = (\phi(\partial_B c))^\dagger$$

for all $a \in \Gamma(A)$, $\phi \in \Gamma(\text{Hom}(B, C))$ and $c \in \Gamma(C)$. To see this, write ϕ as $\sum f_{ij} \cdot \beta_i \cdot c_j$ with $f_{ij} \in C^\infty(M)$, $\beta_i \in \Gamma(B^*)$ and $c_j \in \Gamma(C)$. Then $\widetilde{\phi} = \sum q_B^* f_{ij} \cdot \ell_{\beta_i} \cdot c_j^\dagger$ and one can use the formulas in Theorem 2.8 and the Leibniz rule to compute the brackets with $\sigma_A(a)$ and c^\dagger .

Note that (9) and (10) can also be proved by diagrammatic methods.

Example 2.9. Choose a linear connection $\nabla: \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ and consider the corresponding linear splitting Σ^∇ of TE as in Section 2.1.2. The description of the Lie bracket of vector fields in (4) shows that the 2-representation induced by Σ^∇ is the 2-representation of TM on $\text{Id}_E: E \rightarrow E$ given by $(\nabla, \nabla, R_\nabla)$.

Remark 2.10. If $\Sigma_1, \Sigma_2: A \times_M B \rightarrow D$ are two linear splittings of a VB-algebroid $(D \rightarrow B, A \rightarrow M)$, then the two corresponding 2-representations are related by the following identities [5].

$$\nabla_a^{B,2} = \nabla_a^{B,1} + \partial_B \circ \Phi_{12}(a), \quad \nabla_a^{C,2} = \nabla_a^{C,1} + \Phi_{12}(a) \circ \partial_B$$

and

$$R^2(a_1, a_2) = R^1(a_1, a_2) + (\mathbf{d}_{\nabla^{\text{Hom}(B, C)}} \Phi_{12})(a_1, a_2) + \Phi_{12}(a_1) \partial_B \Phi_{12}(a_2) - \Phi_{12}(a_2) \partial_B \Phi_{12}(a_1)$$

for all $a, a_1, a_2 \in \Gamma(A)$.

2.3.1. Dualization and 2-representations. Let $(D; A, B; M)$ be a VB-algebroid with Lie algebroid structures on $D \rightarrow B$ and $A \rightarrow M$. Let $\Sigma: A \times_M B \rightarrow D$ be a linear splitting of D and denote by (∇^B, ∇^C, R) the 2-representation of the Lie algebroid A on $\partial_B: C \rightarrow B$. We have seen above that $(D \star A; A, C^*; M)$ has an induced VB-algebroid structure, and we have shown that the linear splitting Σ induces a natural linear splitting $\Sigma^*: A \times_M C^* \rightarrow D \star B$ of $D \star A$. The 2-representation of A that is associated to this splitting is then $(\nabla^{C^*}, \nabla^{B^*}, -R^*)$ on the complex $\partial_B^*: B^* \rightarrow C^*$. This is easy to verify, and proved in the appendix⁴ of [2]. One only needs to recall for the proof that, by construction, $\ell_{\sigma_A^*(a)}$ equals $\ell_{\sigma_A(a)}$ as a function on $D \star B$.

2.3.2. The tangent of a Lie algebroid. Let $(A \rightarrow M, \rho, [\cdot, \cdot])$ be a Lie algebroid. Then the tangent $TA \rightarrow TM$ has a Lie algebroid structure with bracket defined by $[Ta_1, Ta_2] = T[a_1, a_2]$, $[Ta_1, a_2^\dagger] = [a_1, a_2]^\dagger$ and $[a_1^\dagger, a_2^\dagger] = 0$ for all $a_1, a_2 \in \Gamma(A)$. The anchor of Ta is $[\rho(a), \cdot] \in \mathfrak{X}(TM)$ and the anchor of a^\dagger is $\rho(a)^\dagger$ for all $a \in \Gamma(A)$. This defines a VB-algebroid structure $(TA \rightarrow TM; A \rightarrow M)$ on $(TA; TM, A; M)$.

Given a TM -connection on A , and so a linear splitting Σ^∇ of TA as in Section 2.1.2, the 2-representation of A on $\rho: A \rightarrow TM$ encoding the VB-algebroid is $(\nabla^{\text{bas}}, \nabla^{\text{bas}}, R^{\text{bas}})$, where the connections are defined by

$$\begin{aligned} \nabla^{\text{bas}}: \Gamma(A) \times \mathfrak{X}(M) &\rightarrow \mathfrak{X}(M), \\ \nabla_a^{\text{bas}} X &= [\rho(a), X] + \rho(\nabla_X a) \end{aligned}$$

and

$$\begin{aligned} \nabla^{\text{bas}}: \Gamma(A) \times \Gamma(A) &\rightarrow \Gamma(A), \\ \nabla_{a_1}^{\text{bas}} a_2 &= [a_1, a_2] + \nabla_{\rho(a_2)} a_1, \end{aligned}$$

and $R_{\nabla}^{\text{bas}} \in \Omega^2(A, \text{Hom}(TM, A))$ is given by

$$R_{\nabla}^{\text{bas}}(a_1, a_2)X = -\nabla_X[a_1, a_2] + [\nabla_X a_1, a_2] + [a_1, \nabla_X a_2] + \nabla_{\nabla_{a_2}^{\text{bas}} X} a_1 - \nabla_{\nabla_{a_1}^{\text{bas}} X} a_2$$

for all $X \in \mathfrak{X}(M)$, $a, a_1, a_2 \in \Gamma(A)$.

3. MAIN THEOREM AND EXAMPLES

We define in this section the notion of *matched pair of representations up to homotopy* — as above, we consider only representations up to homotopy which are concentrated in degrees 0 and 1; that is, 2-representations. We then state our main result: a double vector bundle endowed with two VB-algebroid structures (on each of its sides) is a double Lie algebroid if and only if, for each linear splitting, the two induced representations up to homotopy form a matched pair.

In the second part of this section, we work out the example of the tangent double of a Lie algebroid.

⁴The construction of the “dual” linear splitting of $D \star A$, given a linear splitting of D , is done in [2] by dualizing the decomposition and taking its inverse. The resulting linear splitting of $D \star A$ is the same.

3.1. Matched pairs of representations up to homotopy and main result.

Definition 3.1. Let $(A \rightarrow M, \rho_A, [\cdot, \cdot])$ and $(B \rightarrow M, \rho_B, [\cdot, \cdot])$ be two Lie algebroids and assume that A acts on $\partial_B: C \rightarrow B$ up to homotopy via $(\nabla^{AB}, \nabla^{AC}, R_A)$ and B acts on $\partial_A: C \rightarrow A$ up to homotopy via $(\nabla^{BA}, \nabla^{BC}, R_B)$. Then we say that the two 2-representations form a matched pair if the following hold:⁵

- (M1) $\rho_A \circ \partial_A = \rho_B \circ \partial_B$,
- (M2) $\nabla_{\partial_A c_1} c_2 - \nabla_{\partial_B c_2} c_1 = -\nabla_{\partial_A c_2} c_1 + \nabla_{\partial_B c_1} c_2$,
- (M3) $[a, \partial_A c] = \partial_A(\nabla_a c) - \nabla_{\partial_B c} a$,
- (M4) $[b, \partial_B c] = \partial_B(\nabla_b c) - \nabla_{\partial_A c} b$,
- (M5) $[\rho_A(a), \rho_B(b)] = \rho_B(\nabla_a b) - \rho_A(\nabla_b a)$,
- (M6) $\nabla_b \nabla_a c - \nabla_a \nabla_b c - \nabla_{\nabla_b a} c + \nabla_{\nabla_a b} c = R_B(b, \partial_B c)a - R_A(a, \partial_A c)b$,
- (M7) $\partial_A(R_A(a_1, a_2)b) = -\nabla_b[a_1, a_2] + [\nabla_b a_1, a_2] + [a_1, \nabla_b a_2] + \nabla_{\nabla_{a_2} b} a_1 - \nabla_{\nabla_{a_1} b} a_2$,
- (M8) $\partial_B(R_B(b_1, b_2)a) = -\nabla_a[b_1, b_2] + [\nabla_a b_1, b_2] + [b_1, \nabla_a b_2] + \nabla_{\nabla_{b_2} a} b_1 - \nabla_{\nabla_{b_1} a} b_2$,

for all $a, a_1, a_2 \in \Gamma(A)$, $b, b_1, b_2 \in \Gamma(B)$ and $c, c_1, c_2 \in \Gamma(C)$, and

- (M9) $\mathbf{d}_{\nabla^A} R_B = \mathbf{d}_{\nabla^B} R_A \in \Omega^2(A, \wedge^2 B^* \otimes C) = \Omega^2(B, \wedge^2 A^* \otimes C)$, where R_B is seen as an element of $\Omega^1(A, \wedge^2 B^* \otimes C)$ and R_A as an element of $\Omega^1(B, \wedge^2 A^* \otimes C)$.

Remark 3.2. (1) Compare equations (M1) to (M9) with equations (50) to (58) of [20].
 (2) Note that if C is trivial, then $\partial_A, \partial_B, R_A, R_B$ and ∇^{AC}, ∇^{BC} are trivial. In that case, equations (M1)–(M4), (M6) and (M9) and the left hand sides of (M7) and (M8) vanish. We find hence the definition of a matched pair of representations of Lie algebroids [16, 8].

Remark 3.3. Note that the vector bundle C inherits a Lie algebroid structure with anchor $\rho_C := \rho_A \circ \partial_A = \rho_B \circ \partial_B$ and with bracket given by $[c_1, c_2] = \nabla_{\partial_A c_1} c_2 - \nabla_{\partial_B c_2} c_1$ for all $c_1, c_2 \in \Gamma(C)$.

The choice of sign is the natural one for the Leibniz identity to be satisfied. The proof of the Jacobi identity is not completely straightforward; it follows from (M3), (M4) and (M6). A detailed proof of a more general result, but with the same type of computation, is given in [6, Theorem 6.12]. Note that (M3) together with $\partial_A \circ \nabla^{BC} = \nabla^{BA} \circ \partial_A$ (Equation (2) in the definition of a 2-representation) shows that $\partial_A: C \rightarrow A$ is a Lie algebroid morphism. In the same manner, (M4) together with $\partial_B \circ \nabla^{AC} = \nabla^{AB} \circ \partial_B$ shows that $\partial_B: C \rightarrow B$ is a Lie algebroid morphism.

Theorem 3.4 is our main result. The proof is in §4.

Theorem 3.4. Let $(D; A, B; M)$ be a double vector bundle with VB-algebroid structures on both $(D \rightarrow A, B \rightarrow M)$ and $(D \rightarrow B, A \rightarrow M)$. Choose a linear splitting Σ of D and let \mathcal{D}_A and \mathcal{D}_B be the two 2-representations defined by the lifts σ_A and σ_B . Then $(D; A, B; M)$ is a double Lie algebroid if and only if the two 2-representations form a matched pair.

It is easy to see using Remark 2.10 that the induced Lie algebroid structure on the core C of the double Lie algebroid does not depend on the choice of splitting.

Remark 3.5. Given a matched pair of representations of Lie algebroids A and B on the same base M , the direct sum vector bundle $A \oplus B$ has a Lie algebroid structure, the *bicrossproduct Lie algebroid*, denoted $A \bowtie B$ [8, 16]. The matched pair structure also induces on the decomposed double vector bundle $A \times_M B$ a double Lie algebroid structure and, conversely, any vacant double Lie algebroid (that is, a double Lie algebroid for which the core is zero) arises from a matched pair of Lie algebroids in this way [14, §6].

⁵For the sake of simplicity, from now on we usually write ∇ for all four connections. It is always clear from the indexes which connection is meant. We write ∇^A for the A -connection induced by ∇^{AB} and ∇^{AC} on $\wedge^2 B^* \otimes C$ and ∇^B for the B -connection induced on $\wedge^2 A^* \otimes C$.

3.2. The tangent double of a Lie algebroid. Let $A \rightarrow M$ be a Lie algebroid with anchor ρ . We have seen in Section 2.3.2 that

$$\begin{array}{ccc} TA & \xrightarrow{p_A} & A \\ Tq_A \downarrow & & \downarrow q_A \\ TM & \xrightarrow{p_M} & M \end{array}$$

is endowed with two VB-algebroid structures. $(TA \rightarrow A, TM \rightarrow M)$ has the standard tangent bundle Lie algebroid structure (Example 2.5) and $(TA \rightarrow TM, A \rightarrow M)$ is the tangent prolongation of $A \rightarrow M$ (Section 2.3.2).

Recall from Section 2.1.2 that a linear connection $\nabla: \mathfrak{X}(M) \times \Gamma(A) \rightarrow \Gamma(A)$ defines a linear splitting $\Sigma^\nabla: A \times_M TM \rightarrow TA$. This linear splitting induces the two following 2-representations:

- (1) The VB-algebroid $(TA \rightarrow A, TM \rightarrow M)$ is described by the 2-representation of TM on $\text{Id}_A: A \rightarrow A$ via $(\nabla, \nabla, R_\nabla)$ (Example 2.9). The anchor of TM is Id_{TM} and the bracket is the Lie bracket of vector fields.
- (2) The VB-algebroid $(TA \rightarrow TM, A \rightarrow M)$ is described by the 2-representation of A on $\rho: A \rightarrow TM$ via $(\nabla^{\text{bas}}, \nabla^{\text{bas}}, R_\nabla^{\text{bas}})$ (Section 2.3.2).

We check that these two 2-representations form a matched pair. This will provide a new proof of the fact that the tangent double of a Lie algebroid is a double Lie algebroid [14]. Condition (M1) in Definition 3.1 is immediate, (M2) and (M3) are just two times the definition of $\nabla^{\text{bas}}: \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$ and (M4) and (M5) are two times the definition of $\nabla^{\text{bas}}: \Gamma(A) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$. Condition (M7) is the definition of R_∇^{bas} . Hence, we only have to check (M6), (M8) and (M9).

In the following, X, X_1, X_2 will be arbitrary vector fields on M and a, a_1, a_2 arbitrary sections of A .

(M6) The left-hand side of (M6) is

$$\begin{aligned} \nabla_X \nabla_{a_1}^{\text{bas}} a_2 - \nabla_{a_1}^{\text{bas}} \nabla_X a_2 - \nabla_{\nabla_X a_1}^{\text{bas}} a_2 + \nabla_{\nabla_{a_1}^{\text{bas}} X} a_2 \\ = \nabla_X [a_1, a_2] + \nabla_X \nabla_{\rho(a_2)} a_1 - [a_1, \nabla_X a_2] - \nabla_{\rho(\nabla_X a_2)} a_1 \\ - [\nabla_X a_1, a_2] - \nabla_{\rho(a_2)} \nabla_X a_1 + \nabla_{[\rho(a_1), X]} a_2 + \nabla_{\rho(\nabla_X a_1)} a_2. \end{aligned}$$

The second and sixth term add up to $R(X, \rho(a_2))a_1 + \nabla_{[X, \rho(a_2)]} a_1$ and the first, third, and fifth term to $-R_\nabla^{\text{bas}}(a_1, a_2)X + \nabla_{\nabla_{a_2}^{\text{bas}} X} a_1 - \nabla_{\nabla_{a_1}^{\text{bas}} X} a_2$. The definition of $\nabla^{\text{bas}}: \Gamma(A) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ yields then immediately the right hand side of (M6), namely

$$R(X, \rho(a_2))a_1 - R_\nabla^{\text{bas}}(a_1, a_2)X.$$

(M8) This equation is easily verified:

$$\begin{aligned} -\nabla_a^{\text{bas}} [X_1, X_2] + [\nabla_a^{\text{bas}} X_1, X_2] + [X_1, \nabla_a^{\text{bas}} X_2] + \nabla_{\nabla_{X_2} a}^{\text{bas}} X_1 - \nabla_{\nabla_{X_1} a}^{\text{bas}} X_2 \\ = -[\rho(a), [X_1, X_2]] - \rho(\nabla_{[X_1, X_2]} a) + [[\rho(a), X_1] + \rho(\nabla_{X_1} a), X_2] \\ + [X_1, [\rho(a), X_2] + \rho(\nabla_{X_2} a)] + [\rho(\nabla_{X_2} a), X_1] \\ + \rho(\nabla_{X_1} \nabla_{X_2} a) - [\rho(\nabla_{X_1} a), X_2] - \rho(\nabla_{X_2} \nabla_{X_1} a) \\ = \rho(R_\nabla(X_1, X_2)a) \end{aligned}$$

To get the second equality, we use the Jacobi identity for the Lie bracket of vector fields. (The four remaining terms cancel pairwise).

(M9) As one would expect, checking (M9) is a long, but straightforward computation. We carry this out in detail here, because we will not give all the details of the proof of Theorem

3.4. We begin by computing

$$\begin{aligned}
(\mathbf{d}_\nabla R_\nabla^{\text{bas}})(X_1, X_2)(a_1, a_2) &= -R_\nabla^{\text{bas}}(a_1, a_2)[X_1, X_2] \\
&\quad + \nabla_{X_1}(R_\nabla^{\text{bas}}(a_1, a_2)X_2) - \nabla_{X_2}(R_\nabla^{\text{bas}}(a_1, a_2)X_1) \\
&\quad - R_\nabla^{\text{bas}}(\nabla_{X_1}a_1, a_2)X_2 - R_\nabla^{\text{bas}}(a_1, \nabla_{X_1}a_2)X_2 \\
&\quad + R_\nabla^{\text{bas}}(\nabla_{X_2}a_1, a_2)X_1 + R_\nabla^{\text{bas}}(a_1, \nabla_{X_2}a_2)X_1.
\end{aligned}$$

This expands out to

$$\begin{aligned}
&\nabla_{[X_1, X_2]}[a_1, a_2] - [\nabla_{[X_1, X_2]}a_1, a_2] - [a_1, \nabla_{[X_1, X_2]}a_2] + \nabla_{\nabla_{a_1}^{\text{bas}}[X_1, X_2]}a_2 - \nabla_{\nabla_{a_2}^{\text{bas}}[X_1, X_2]}a_1 \\
&\quad + \nabla_{X_1} \left(-\nabla_{X_2}[a_1, a_2] + \cancel{[\nabla_{X_2}a_1, a_2]} + \cancel{[a_1, \nabla_{X_2}a_2]} - \nabla_{\nabla_{a_1}^{\text{bas}}X_2}a_2 + \nabla_{\nabla_{a_2}^{\text{bas}}X_2}a_1 \right) \\
&\quad - \nabla_{X_2} \left(-\nabla_{X_1}[a_1, a_2] + \cancel{[\nabla_{X_1}a_1, a_2]} + \cancel{[a_1, \nabla_{X_1}a_2]} - \nabla_{\nabla_{a_1}^{\text{bas}}X_1}a_2 + \nabla_{\nabla_{a_2}^{\text{bas}}X_1}a_1 \right) \\
&\quad + \cancel{\nabla_{X_2}[\nabla_{X_1}a_1, a_2]} - [\nabla_{X_2}\nabla_{X_1}a_1, a_2] - \cancel{[\nabla_{X_1}a_1, \nabla_{X_2}a_2]} + \nabla_{\nabla_{X_1}^{\text{bas}}X_2}a_2 - \nabla_{\nabla_{X_2}^{\text{bas}}X_1}a_1 \\
&\quad + \cancel{\nabla_{X_2}[a_1, \nabla_{X_1}a_2]} - \cancel{[\nabla_{X_2}a_1, \nabla_{X_1}a_2]} - [a_1, \nabla_{X_2}\nabla_{X_1}a_2] + \nabla_{\nabla_{a_1}^{\text{bas}}X_2}\nabla_{X_1}a_2 - \nabla_{\nabla_{X_1}^{\text{bas}}X_2}a_1 \\
&\quad - \cancel{\nabla_{X_1}[\nabla_{X_2}a_1, a_2]} + [\nabla_{X_1}\nabla_{X_2}a_1, a_2] + \cancel{[\nabla_{X_2}a_1, \nabla_{X_1}a_2]} - \nabla_{\nabla_{X_2}^{\text{bas}}X_1}a_2 + \nabla_{\nabla_{a_2}^{\text{bas}}X_1}\nabla_{X_2}a_1 \\
&\quad - \cancel{\nabla_{X_1}[a_1, \nabla_{X_2}a_2]} + \cancel{[\nabla_{X_1}a_1, \nabla_{X_2}a_2]} + [a_1, \nabla_{X_1}\nabla_{X_2}a_2] - \nabla_{\nabla_{a_1}^{\text{bas}}X_1}\nabla_{X_2}a_2 + \nabla_{\nabla_{X_2}^{\text{bas}}X_1}a_1.
\end{aligned}$$

Twelve terms of this equation cancel pairwise as shown, and a reordering of the remaining terms yields

$$\begin{aligned}
&-R_\nabla(X_1, X_2)[a_1, a_2] + [R_\nabla(X_1, X_2)a_1, a_2] + [a_1, R_\nabla(X_1, X_2)a_2] \\
&\quad + R_\nabla(X_2, \nabla_{a_1}^{\text{bas}}X_1)a_2 + \nabla_{[X_2, \nabla_{a_1}^{\text{bas}}X_1]}a_2 - R_\nabla(X_2, \nabla_{a_2}^{\text{bas}}X_1)a_1 - \nabla_{[X_2, \nabla_{a_2}^{\text{bas}}X_1]}a_1 \\
&\quad - R_\nabla(X_1, \nabla_{a_1}^{\text{bas}}X_2)a_2 - \nabla_{[X_1, \nabla_{a_1}^{\text{bas}}X_2]}a_2 + R_\nabla(X_1, \nabla_{a_2}^{\text{bas}}X_2)a_1 + \nabla_{[X_1, \nabla_{a_2}^{\text{bas}}X_2]}a_1 \\
&\quad + \nabla_{\nabla_{a_1}^{\text{bas}}[X_1, X_2]}a_2 - \nabla_{\nabla_{a_2}^{\text{bas}}[X_1, X_2]}a_1 - \nabla_{\nabla_{X_2}^{\text{bas}}a_1}X_1a_2 \\
&\quad + \nabla_{\nabla_{X_2}^{\text{bas}}a_2}X_1a_1 + \nabla_{\nabla_{X_1}^{\text{bas}}a_1}X_2a_2 - \nabla_{\nabla_{X_1}^{\text{bas}}a_2}X_2a_1.
\end{aligned}$$

By (M8), this equals

$$\begin{aligned}
&-R_\nabla(X_1, X_2)[a_1, a_2] + [R_\nabla(X_1, X_2)a_1, a_2] + [a_1, R_\nabla(X_1, X_2)a_2] \\
&\quad + R_\nabla(X_2, \nabla_{a_1}^{\text{bas}}X_1)a_2 - R_\nabla(X_2, \nabla_{a_2}^{\text{bas}}X_1)a_1 - R_\nabla(X_1, \nabla_{a_1}^{\text{bas}}X_2)a_2 + R_\nabla(X_1, \nabla_{a_2}^{\text{bas}}X_2)a_1 \\
&\quad + \nabla_{\rho(R_\nabla(X_1, X_2)a_2)}a_1 - \nabla_{\rho(R_\nabla(X_1, X_2)a_1)}a_2,
\end{aligned}$$

which is

$$\begin{aligned}
&-R_\nabla(X_1, X_2)[a_1, a_2] - \nabla_{a_2}^{\text{bas}}R_\nabla(X_1, X_2)a_1 + \nabla_{a_1}^{\text{bas}}R_\nabla(X_1, X_2)a_2 \\
&\quad - R_\nabla(\nabla_{a_1}^{\text{bas}}X_1, X_2)a_2 + R_\nabla(\nabla_{a_2}^{\text{bas}}X_1, X_2)a_1 - R_\nabla(X_1, \nabla_{a_1}^{\text{bas}}X_2)a_2 + R_\nabla(X_1, \nabla_{a_2}^{\text{bas}}X_2)a_1 \\
&= (\mathbf{d}_{\nabla^{\text{bas}}}R_\nabla)(a_1, a_2)(X_1, X_2).
\end{aligned}$$

4. PROOF OF THE THEOREM

We will prove the theorem by checking the Lie bialgebroid condition only on particular families of sections; the linear sections and the core sections. The main difficulty is to understand the additional conditions which have to be verified by the families of sections for the proof to be complete. This is done in Subsection 4.1. In Subsection 4.2, we will show how the equations found in Subsection 4.1 imply (M1)–(M9) and vice-versa.

4.1. Families of sections of Lie bialgebroids. We recall the definition of a Lie bialgebroid [15]; see also [13, Chapter 12]. We will then show how the equation defining a Lie bialgebroid (A, A^*) can be verified only on families of spanning sections of A and A^* .

Definition 4.1. *Let $q_A: A \rightarrow M$ and $q_{A^*}: A^* \rightarrow M$ be a pair of dual vector bundles, and suppose each has a Lie algebroid structure, with anchors $\rho: A \rightarrow TM$ and $\rho_*: A^* \rightarrow TM$ respectively, and brackets $[\cdot, \cdot]$ and $[\cdot, \cdot]_*$.*

Then (A, A^) is a Lie bialgebroid if for all $a_1, a_2 \in \Gamma(A)$:*

$$(11) \quad \mathbf{d}_{A^*}[a_1, a_2] = [\mathbf{d}_{A^*}a_1, a_2] + [a_1, \mathbf{d}_{A^*}a_2].$$

The brackets on the RHS are extensions to 2-vectors by standard Schouten calculus.

It is often very convenient to check this condition only on the elements of a given set of sections $\mathcal{S} \subseteq \Gamma(A)$ which spans $\Gamma(A)$ as a $C^\infty(M)$ -module. We will formalize this technique shortly. We first need to recall some consequences of the definition.

The proof of the following proposition is a straightforward computation.

Proposition 4.2. *Let A and A^* be dual vector bundles with Lie algebroid structures. For $a_1, a_2 \in \Gamma(A)$, $\alpha_1, \alpha_2 \in \Gamma(A^*)$ and $f \in C^\infty(M)$, we have*

$$(12) \quad \begin{aligned} & (\mathbf{d}_{A^*}[a_1, fa_2] - [\mathbf{d}_{A^*}a_1, fa_2] - [a_1, \mathbf{d}_{A^*}(fa_2)])(\alpha_1, \alpha_2) \\ &= f \cdot (\mathbf{d}_{A^*}[a_1, a_2] - [\mathbf{d}_{A^*}a_1, a_2] - [a_1, \mathbf{d}_{A^*}a_2])(\alpha_1, \alpha_2) \\ & \quad - \langle a_2, \alpha_2 \rangle \cdot ([\rho(a_1), \rho_*(\alpha_1)](f) - \rho_*(\mathcal{L}_{a_1}\alpha_1)(f) + \rho(\mathcal{L}_{\alpha_1}a_1)(f) - \rho_*(\mathbf{d}_A f)\langle a_1, \alpha_1 \rangle) \\ & \quad + \langle a_2, \alpha_1 \rangle \cdot ([\rho(a_1), \rho_*(\alpha_2)](f) - \rho_*(\mathcal{L}_{a_1}\alpha_2)(f) + \rho(\mathcal{L}_{\alpha_2}a_1)(f) - \rho_*(\mathbf{d}_A f)\langle a_1, \alpha_2 \rangle). \end{aligned}$$

Now assume that (A, A^*) is a Lie bialgebroid. Take any $a_1 \in \Gamma A$ and any nonvanishing $\alpha_1 \in \Gamma(A^*)$. Choose a nonvanishing $a_2 \in \Gamma(A)$ and an $\alpha_2 \in \Gamma(A^*)$ such that $\langle a_2, \alpha_1 \rangle = 0$ and $\langle a_2, \alpha_2 \rangle = 1$. (If A has rank 1 then (13) below is vacuously true.) Equation (12) now reduces to

$$(13) \quad [\rho(a_1), \rho_*(\alpha_1)](f) - \rho_*(\mathcal{L}_{a_1}\alpha_1)(f) + \rho(\mathcal{L}_{\alpha_1}a_1)(f) - \rho_*(\mathbf{d}_A f)\langle a_1, \alpha_1 \rangle = 0$$

for all $a_1 \in \Gamma(A)$, $f \in C^\infty(M)$, and nonvanishing $\alpha_1 \in \Gamma(A^*)$. A straightforward computation shows that the left-hand side of (13) is tensorial in the term α_1 . Hence, (13) holds for all $\alpha_1 \in \Gamma(A^*)$. (For another proof, see [13, 12.1.8].)

On the other hand, the left-hand side of (13) is not tensorial in the term a_1 . We multiply a_1 by a function $g \in C^\infty(M)$ in this equation, expand out, and subtract

$$g \cdot ([\rho(a_1), \rho_*(\alpha_1)](f) - \rho_*(\mathcal{L}_{a_1}\alpha_1)(f) + \rho(\mathcal{L}_{\alpha_1}a_1)(f) - \rho_*(\mathbf{d}_A f)\langle a_1, \alpha_1 \rangle) = 0.$$

We get that

$$\langle a_1, \alpha_1 \rangle \cdot (-\rho_*(\mathbf{d}_A g)(f) - \rho_*(\mathbf{d}_A f)(g)) = 0.$$

Again, since a_1 and α_1 were arbitrary, we have found

$$-\rho_*(\mathbf{d}_A g)(f) = \rho_*(\mathbf{d}_A f)(g) \quad \text{for all } f, g \in C^\infty(M),$$

which is easily seen to be equivalent to

$$(14) \quad -\rho \circ \rho_*^* = \rho_* \circ \rho^*,$$

see also [15], [13, §12.1]. The map $\rho_* \circ \rho^*: T^*M \rightarrow TM$ defines a Poisson structure on M , which we take to be the *Poisson structure on M induced by the Lie bialgebroid structure*.

These considerations lead to the following result.

Proposition 4.3. *Let $q_A: A \rightarrow M$ and $q_{A^*}: A^* \rightarrow M$ be a pair of dual vector bundles, and suppose each has a Lie algebroid structure, with anchors $\rho: A \rightarrow TM$ and $\rho_*: A^* \rightarrow TM$ respectively, and brackets $[\cdot, \cdot]$ and $[\cdot, \cdot]_*$. Let \mathcal{S} be a subset of $\Gamma(A)$ which spans $\Gamma(A)$ as a $C^\infty(M)$ -module.*

Then (A, A^) is a Lie bialgebroid if and only if the following three conditions hold.*

- (B1) $\mathbf{d}_{A^*}[a_1, a_2] = [\mathbf{d}_{A^*}a_1, a_2] + [a_1, \mathbf{d}_{A^*}a_2]$ for all $a_1, a_2 \in \mathcal{S}$,
 (B2) $[\rho(a), \rho_*(\alpha)](f) - \rho_*(\mathcal{L}_a\alpha)(f) + \rho(\mathcal{L}_a\alpha)(f) - \rho_*(\mathbf{d}_A f)\langle a, \alpha \rangle = 0$ for all $a \in \mathcal{S}$, $\alpha \in \Gamma(A^*)$ and $f \in C^\infty(M)$, and
 (B3) $-\rho \circ \rho_*^* = \rho_* \circ \rho^*$.

Proof. We proved above that these three conditions hold when (A, A^*) is a Lie bialgebroid. For the converse, a quick computation using (B1) and the considerations before the proposition shows that

$$\begin{aligned} & (\mathbf{d}_{A^*}[ga_1, fa_2] - [\mathbf{d}_{A^*}(ga_1), fa_2] - [ga_1, \mathbf{d}_{A^*}(fa_2)])(\alpha_1, \alpha_2) \\ &= fg \cdot (\mathbf{d}_{A^*}[a_1, a_2] - [\mathbf{d}_{A^*}a_1, a_2] - [a_1, \mathbf{d}_{A^*}a_2])(\alpha_1, \alpha_2) \\ & \quad - fg\langle a_2, \alpha_2 \rangle \cdot ([\rho(a_1), \rho_*(\alpha_1)](f) - \rho_*(\mathcal{L}_{a_1}\alpha_1)(f) + \rho(\mathcal{L}_{a_1}\alpha_1)(f) - \rho_*(\mathbf{d}_A f)\langle a_1, \alpha_1 \rangle) \\ & \quad + fg\langle a_2, \alpha_1 \rangle \cdot ([\rho(a_1), \rho_*(\alpha_2)](f) - \rho_*(\mathcal{L}_{a_1}\alpha_2)(f) + \rho(\mathcal{L}_{a_1}\alpha_2)(f) - \rho_*(\mathbf{d}_A f)\langle a_1, \alpha_2 \rangle) \\ & \quad + fg\langle a_1, \alpha_1 \rangle \cdot ([\rho(a_2), \rho_*(\alpha_2)](f) - \rho_*(\mathcal{L}_{a_2}\alpha_2)(f) + \rho(\mathcal{L}_{a_2}\alpha_2)(f) - \rho_*(\mathbf{d}_A f)\langle a_2, \alpha_2 \rangle) \\ & \quad - fg\langle a_1, \alpha_2 \rangle \cdot ([\rho(a_2), \rho_*(\alpha_1)](f) - \rho_*(\mathcal{L}_{a_2}\alpha_1)(f) + \rho(\mathcal{L}_{a_2}\alpha_1)(f) - \rho_*(\mathbf{d}_A f)\langle a_2, \alpha_1 \rangle). \end{aligned}$$

for all $a_1, a_2 \in \mathcal{S}$, $\alpha_1, \alpha_2 \in \Gamma(A^*)$ and $f, g \in C^\infty(M)$. This vanishes by (B1) and (B2). Since the Lie bialgebroid condition is additive and $\Gamma(A)$ is spanned as a $C^\infty(M)$ -module by \mathcal{S} , we are done. \square

Remark 4.4. In Proposition 4.3, the first two conditions are $C^\infty(M)$ -linear in the $\Gamma(A^*)$ -argument, so it is sufficient to check them on a subset $\mathcal{R} \subseteq \Gamma(A^*)$ that spans $\Gamma(A^*)$ as a $C^\infty(M)$ -module.

4.2. The Lie bialgebroid conditions on lifts and on core sections. We write here $\Theta_A: D \star A \rightarrow TC^*$ for the anchor of $D \star A \rightarrow C^*$ and $\Theta_B: D \star B \rightarrow TC^*$ for the anchor of $D \star B \rightarrow C^*$. We set

$$\mathcal{S} := \Gamma_{C^*}^c(D \star A) \cup \sigma_A^*(\Gamma(A))$$

and

$$\mathcal{R} := \Gamma_{C^*}^c(D \star B) \cup \sigma_B^*(\Gamma(B)).$$

Proposition 4.5. Condition (B3) on \mathcal{S} and \mathcal{R} is equivalent to (M1) and (M2).

Proof. Since

$$\Theta_A \circ \Theta_B^*, \Theta_B \circ \Theta_A^*: T^*C^* \rightarrow TC^*.$$

are vector bundle maps, it is sufficient to check (B3) on $\mathbf{d}F$ for $F \in C^\infty(C^*)$. In fact, it is even sufficient to check (B3) on $\mathbf{d}(q_{C^*}^* f)$ for $f \in C^\infty(M)$ and $\mathbf{d}\ell_c$ for $c \in \Gamma(C)$.

Choose first $f \in C^\infty(M)$ and consider $q_{C^*}^* f \in C^\infty(C^*)$. We have for any section $b \in \Gamma(B)$:

$$\langle \Theta_B^*(\mathbf{d}q_{C^*}^* f), \sigma_B^*(b) \rangle = \widehat{\nabla}_b^*(q_{C^*}^* f) = q_{C^*}^*(\rho_B(b)f)$$

and for any $\alpha \in \Gamma(A^*)$:

$$\langle \Theta_B^*(\mathbf{d}q_{C^*}^* f), \alpha^\dagger \rangle = (\partial_A^* \alpha)^\dagger(q_{C^*}^* f) = 0.$$

This shows

$$(15) \quad \Theta_B^*(\mathbf{d}q_{C^*}^* f) = (\rho_B^* \mathbf{d}f)^\dagger \in \Gamma_{C^*}^c(D \star A).$$

We get consequently $\Theta_A \circ \Theta_B^*(\mathbf{d}q_{C^*}^* f) = (\partial_B^* \rho_B^* \mathbf{d}f)^\dagger \in \mathfrak{X}(C^*)$. In the same manner, we find $\Theta_A \circ \Theta_B^*(\mathbf{d}q_{C^*}^* f) = (-\partial_A^* \rho_A^* \mathbf{d}f)^\dagger \in \mathfrak{X}(C^*)$. The equality of $\Theta_A \circ \Theta_B^*$ and $-\Theta_B \circ \Theta_A^*$ on pullbacks is hence equivalent to $\rho_A \circ \partial_A = \rho_B \circ \partial_B$.

We continue with linear functions. Choose $c \in \Gamma(C)$. Then for any section $b \in \Gamma(B)$, we get

$$\langle \Theta_B^*(\mathbf{d}\ell_c), \sigma_B^*(b) \rangle = \widehat{\nabla}_b^*(\ell_c) = \ell_{\nabla_b c}$$

and for any $\alpha \in \Gamma(A^*)$:

$$\langle \Theta_B^*(\mathbf{d}\ell_c), \alpha^\dagger \rangle = (\partial_A^* \alpha)^\dagger(\ell_c) = q_{C^*}^* \langle \alpha, \partial_A c \rangle.$$

This shows

$$(16) \quad \Theta_B^*(\mathbf{d}\ell_c) = -\sigma_A^*(\partial_A c) + \widetilde{\langle \nabla \cdot c, \cdot \rangle} \in \Gamma_{C^*}^l(D \star A),$$

where $\langle \nabla \cdot c, \cdot \rangle$ is seen as an element of $\Gamma(\text{Hom}(C^*, B^*))$. This leads to

$$\Theta_A \circ \Theta_B^*(\mathbf{d}\ell_c)(\ell_{c'}) = -\ell_{\nabla_{\partial_A(c)} c'} + \ell_{\nabla_{\partial_B(c')} c}$$

for all $c' \in \Gamma(C)$ and

$$\Theta_A \circ \Theta_B^*(\mathbf{d}\ell_c)(q_{C^*}^* f) = -q_{C^*}^*(\rho_A \circ \partial_A(c) f)$$

for $f \in C^\infty(M)$. We find similar equations for $\Theta_B \circ \Theta_A^*(\mathbf{d}\ell_c)(\ell_{c'})$ and $\Theta_B \circ \Theta_A^*(\mathbf{d}\ell_c)(q_{C^*}^* f)$, and can conclude that $\Theta_A \circ \Theta_B^* = -\Theta_B \circ \Theta_A^*$ holds if and only if (M1) and (M2) are satisfied. \square

As a corollary of this proof, we find the following result. Recall that the map $\Theta_B \circ \Theta_A^*: T^*C^* \rightarrow TC$ defines a Poisson structure on C^* (see (14) and the considerations following it).

Corollary 4.6. *The Poisson structure on C^* induced by the Lie bialgebroid structure is the linear Poisson structure dual to the Lie algebroid structure on C as in Remark 3.3. More explicitly, it is given by*

$$(17) \quad \begin{aligned} \{\ell_{c_1}, \ell_{c_2}\} &= (\Theta_B \circ \Theta_A^*)(q_{C^*}^* \mathbf{d}\ell_{c_1})(\ell_{c_2}) = \ell_{\nabla_{\partial_A(c_1)}(c_2) - \nabla_{\partial_B(c_2)}(c_1)} = \ell_{[c_1, c_2]}, \\ \{\ell_{c_1}, q_{C^*}^* f\} &= (\Theta_B \circ \Theta_A^*)(q_{C^*}^* \mathbf{d}\ell_{c_1})(q_{C^*}^* f) = q_{C^*}^*(\rho_A(\partial_A(c))(f)) \\ \{q_{C^*}^* f_1, q_{C^*}^* f_2\} &= (\Theta_B \circ \Theta_A^*)(q_{C^*}^* \mathbf{d}f_1)(q_{C^*}^* f) = 0. \end{aligned}$$

Remark 4.7. Note that the apparent asymmetry between the structures over A and B arises from unavoidable choices in the identifications between the various duals. The Poisson structure on C^* is nonetheless determined by requiring ∂_A and ∂_B to be morphisms of Lie algebroids.

For the study of (B1) and (B2), we will need the following lemma. Recall that for a Lie algebroid A , the Lie derivative $\mathcal{L}: \Gamma(A) \times \Gamma(A^*) \rightarrow \Gamma(A^*)$ is defined by

$$\langle \mathcal{L}_a \alpha, a' \rangle = \rho_A(a) \langle \alpha, a' \rangle - \langle \alpha, [a, a'] \rangle$$

for all $a, a' \in \Gamma(A)$ and $\alpha \in \Gamma(A^*)$.

Lemma 4.8. *The Lie derivative $\mathcal{L}: \Gamma_{C^*}(D \star A) \times \Gamma_{C^*}(D \star B) \rightarrow \Gamma_{C^*}(D \star B)$ is given by the following identities:*

$$\begin{aligned} \mathcal{L}_{\beta^\dagger} \alpha^\dagger &= 0, \quad \mathcal{L}_{\beta^\dagger} \sigma_B^*(b) = -\langle b, \nabla^* \beta \rangle^\dagger, \quad \mathcal{L}_{\sigma_A^*(a)} \alpha^\dagger = \mathcal{L}_a \alpha^\dagger, \\ \mathcal{L}_{\sigma_A^*(a)} \sigma_B^*(b) &= \sigma_B^*(\nabla_a b) + \widetilde{R(a, \cdot)} b \end{aligned}$$

for all $a \in \Gamma(A)$, $b \in \Gamma(B)$, $\alpha \in \Gamma(A^*)$ and $\beta \in \Gamma(B^*)$. The Lie derivative $\mathcal{L}: \Gamma_{C^*}(D \star B) \times \Gamma_{C^*}(D \star A) \rightarrow \Gamma_{C^*}(D \star A)$ is given by:

$$\begin{aligned} \mathcal{L}_{\alpha^\dagger} \beta^\dagger &= 0, \quad \mathcal{L}_{\alpha^\dagger} \sigma_A^*(a) = -\langle a, \nabla^* \alpha \rangle^\dagger, \quad \mathcal{L}_{\sigma_A^*(b)} \beta^\dagger = \mathcal{L}_b \beta^\dagger \\ \mathcal{L}_{\sigma_A^*(b)} \sigma_A^*(a) &= \sigma_B^*(\nabla_b a) + \widetilde{R(b, \cdot)} a. \end{aligned}$$

Note that in these equations, $R(a, \cdot)b$ is seen as a section of $\text{Hom}(C^*, A^*)$ and $R(b, \cdot)a$ is seen as a section of $\text{Hom}(C^*, B^*)$.

Proof. We have

$$\begin{aligned}\langle \beta_2^\dagger, \mathcal{L}_{\beta_1^\dagger} \alpha^\dagger \rangle &= (\partial_B^* \beta_1)^\dagger \langle \beta_2^\dagger, \alpha^\dagger \rangle - \langle [\beta_1^\dagger, \beta_2^\dagger], \alpha^\dagger \rangle = 0 \quad \text{and} \\ \langle \sigma_A^*(a), \mathcal{L}_{\beta_1^\dagger} \alpha^\dagger \rangle &= (\partial_B^* \beta_1)^\dagger (-q_{C^*}^* \langle \alpha, a \rangle) + \langle \nabla_a^* \beta_1^\dagger, \alpha^\dagger \rangle = 0\end{aligned}$$

for arbitrary $\beta_1, \beta_2 \in \Gamma(B^*)$, $\alpha \in \Gamma(A^*)$ and $a \in \Gamma(A)$. This proves $\mathcal{L}_{\beta_1^\dagger} \alpha^\dagger = 0$.

Then we compute

$$\langle \beta_2^\dagger, \mathcal{L}_{\beta_1^\dagger} \sigma_B^*(b) \rangle = (\partial_B^* \beta_1)^\dagger (q_{C^*}^* \langle \beta_2, b \rangle) - \langle [\beta_1^\dagger, \beta_2^\dagger], \sigma_B^*(b) \rangle = 0$$

which shows that $\mathcal{L}_{\beta_1^\dagger} \sigma_B^*(b)$ is a section with values in the core, and

$$\langle \sigma_A^*(a), \mathcal{L}_{\beta_1^\dagger} \sigma_B^*(b) \rangle = 0 + \langle \nabla_a^* \beta_1^\dagger, \sigma_B^*(b) \rangle = q_{C^*}^* \langle b, \nabla_a^* \beta_1 \rangle.$$

This proves $\mathcal{L}_{\beta_1^\dagger} \sigma_B^*(b) = -\langle b, \nabla^* \beta_1 \rangle^\dagger$, with $\langle b, \nabla^* \beta_1 \rangle \in \Gamma(A^*)$. We also find

$$\begin{aligned}\langle \beta^\dagger, \mathcal{L}_{\sigma_A^*(a_1)} \alpha^\dagger \rangle &= \widehat{\nabla_{a_1}^*} \langle \beta^\dagger, \alpha^\dagger \rangle - \langle \nabla_{a_1}^* \beta^\dagger, \alpha^\dagger \rangle = 0 \quad \text{and} \\ \langle \sigma_A^*(a_2), \mathcal{L}_{\sigma_A^*(a_1)} \alpha^\dagger \rangle &= -\widehat{\nabla_{a_1}^*} (q_{C^*}^* \langle \alpha, a_2 \rangle) - \langle \sigma_A^*[a_1, a_2] + (R(a_1, a_2)^*)^\dagger, \alpha^\dagger \rangle \\ &= -q_{C^*}^* (\rho_A(a_1) \langle \alpha, a_2 \rangle - \langle \alpha, [a_1, a_2] \rangle) = -q_{C^*}^* \langle \mathcal{L}_{a_1} \alpha, a_2 \rangle\end{aligned}$$

for arbitrary $a_1, a_2 \in \Gamma(A)$ and $\alpha \in \Gamma(A^*)$. This proves the equality $\mathcal{L}_{\sigma_A^*(a_1)} \alpha^\dagger = \mathcal{L}_{a_1} \alpha^\dagger$.

The identity

$$\begin{aligned}\langle \beta^\dagger, \mathcal{L}_{\sigma_A^*(a_1)} \sigma_B^*(b) \rangle &= \widehat{\nabla_{a_1}^*} q_{C^*}^* \langle \beta, b \rangle - \langle \nabla_{a_1}^* \beta^\dagger, \sigma_B^*(b) \rangle \\ &= q_{C^*}^* (\rho_A(a_1) \langle \beta, b \rangle - \langle \nabla_{a_1}^* \beta, b \rangle) = q_{C^*}^* \langle \beta, \nabla_{a_1} b \rangle\end{aligned}$$

shows that $\mathcal{L}_{\sigma_A^*(a_1)} \sigma_B^*(b)$ is the sum of $\sigma_B^*(\nabla_{a_1} b)$ with a section with values in the core. To find out this core term, we finally compute

$$\begin{aligned}\langle \sigma_A^*(a_2), \mathcal{L}_{\sigma_A^*(a_1)} \sigma_B^*(b) \rangle &= 0 - \langle \sigma_A^*[a_1, a_2] + \widetilde{R(a_1, a_2)^*}, \sigma_B^*(b) \rangle \\ &= -\ell_{R(a_1, a_2)(b)}.\end{aligned}$$

This shows that $\mathcal{L}_{\sigma_A^*(a_1)} \sigma_B^*(b) = \sigma_B^*(\nabla_{a_1} b) + \widetilde{R(a_1, \cdot)} b$.

The formulas describing the Lie derivative $\mathcal{L} : \Gamma_{C^*}(D \star B) \times \Gamma_{C^*}(D \star A) \rightarrow \Gamma_{C^*}(D \star A)$ can be verified in the same manner. \square

Proposition 4.9. *Condition (B2) on \mathcal{S} and \mathcal{R} is equivalent to (M3), (M4), (M5) and (M6).*

Proof. The idea of this proof is to check (B2) on linear and core sections in \mathcal{S} and \mathcal{R} , and on linear and q_{C^*} -pullback functions on C^* . We start with core sections. Choose $\alpha \in \Gamma(A^*)$ and $\beta \in \Gamma(B^*)$. We have $[\Theta_B(\alpha^\dagger), \Theta_A(\beta^\dagger)] = [(\partial_A^* \alpha)^\dagger, (\partial_B^* \beta)^\dagger] = 0$. By Lemma 4.8 and with $\langle \beta^\dagger, \alpha^\dagger \rangle = 0$, we find that (B2) is trivially satisfied on $\alpha^\dagger, \beta^\dagger$ and any element of $C^\infty(C^*)$.

Now choose $a \in \Gamma(A)$, $\alpha \in \Gamma(A^*)$. Using Lemma 4.8 we find for all $F \in C^\infty(C^*)$

$$\begin{aligned}[\Theta_B(\alpha^\dagger), \Theta_A(\sigma_A^*(a))](F) &- \Theta_A(\mathcal{L}_{\alpha^\dagger} \sigma_A^*(a))(F) + \Theta_B(\mathcal{L}_{\sigma_A^*(a)} \alpha^\dagger)(F) \\ &- \Theta_A(\mathbf{d}_{D \star B} F) \langle \sigma_A^*(a), \alpha^\dagger \rangle \\ &= -(\nabla_a^* (\partial_A^* \alpha))^\dagger(F) + \langle a, \nabla_{\partial_B^* \alpha}^* \rangle^\dagger(F) + (\partial_A^* \mathcal{L}_a \alpha)^\dagger(F) + (\Theta_A \circ \Theta_B^* \mathbf{d} F) q_{C^*}^* \langle \alpha, a \rangle.\end{aligned}$$

In particular, for $F = q_{C^*}^* f$, $f \in C^\infty(M)$, this is $0 + (\partial_B^* \rho_B^* \mathbf{d} f)^\dagger q_{C^*}^* \langle \alpha, a \rangle = 0$ by (15) and for $F = \ell_c$, $c \in \Gamma(C)$, this is

$$q_{C^*}^* (-\langle \nabla_a^* (\partial_A^* \alpha), c \rangle + \langle a, \nabla_{\partial_B^* c}^* \rangle + \langle \partial_A^* \mathcal{L}_a \alpha, c \rangle - (\rho_B \circ \partial_B(c)) \langle \alpha, a \rangle)$$

by (16). But this equals $q_{C^*}^* (\langle \alpha, \partial_A (\nabla_a c) - \nabla_{\partial_B c}^* a - [a, \partial_A c] \rangle)$. This shows that (B2) is in this case equivalent to (M3). In the same manner, (B2) on $\beta^\dagger \in \mathcal{S}$ for $\beta \in \Gamma(B^*)$, $\sigma_B^*(b) \in \mathcal{R}$ for $b \in \Gamma(B)$ and $F \in C^\infty(C^*)$ is equivalent to (M4).

Now choose $a \in \Gamma(A)$ and $b \in \Gamma(B)$. (B2) on $\sigma_A^*(a)$, $\sigma_B^*(b)$ and $q_{C^*}^* f$, $f \in C^\infty(M)$, is

$$q_{C^*}^* (([\rho_B(b), \rho_A(a)] - \rho_A(\nabla_b a) + \rho_B(\nabla_a b))(f)) = 0$$

by Lemma 4.8. This is (M5). Finally we compute (B2) on $\sigma_A^*(a)$, $\sigma_B^*(b)$ and ℓ_c , for $c \in \Gamma(C)$. This is

$$\ell_{\nabla_b \nabla_a c - \nabla_a \nabla_b c} - \ell_{\nabla_{\nabla_b a} c + R_{BA}(b, \partial_B c)a} + \ell_{\nabla_{\nabla_a b} c + R_{AB}(a, \partial_A c)b} = 0$$

Lemma 4.8. We find hence that (B2) on $\sigma_A^*(a)$, $\sigma_B^*(b)$ and ℓ_c is equivalent to (M6). \square

We conclude the proof of Theorem 3.4 with the study of (B1) on linear and core sections.

Proposition 4.10. *Assume that (M5) is given. Condition (B1) on elements of \mathcal{S} and \mathcal{R} is equivalent to (M7), (M8) and (M9).*

In the proof of this proposition, we will use the following formulas. Let A and A^* be a pair of Lie algebroids in duality. Then, for all $a \in \Gamma(A)$ and $\alpha_1, \alpha_2 \in \Gamma(A^*)$:

$$(\mathbf{d}_{A^*} a)(\alpha_1, \alpha_2) = \rho_{A^*}(\alpha_1)\langle \alpha_2, a \rangle - \rho_{A^*}(\alpha_2)\langle \alpha_1, a \rangle - \langle [\alpha_1, \alpha_2]_{A^*}, a \rangle.$$

For all $a_1, a_2 \in \Gamma(A)$ and $\alpha_1, \alpha_2 \in \Gamma(A^*)$, we have

$$\begin{aligned} [\mathbf{d}_{A^*} a_1, a_2]_A(\alpha_1, \alpha_2) &= -(\mathcal{L}_{a_2} \mathbf{d}_{A^*} a_1)(\alpha_1, \alpha_2) \\ &= -\mathcal{L}_{\rho_A(a_2)}(\mathbf{d}_{A^*} a_1(\alpha_1, \alpha_2)) + \mathbf{d}_{A^*} a_1(\mathcal{L}_{a_2} \alpha_1, \alpha_2) + \mathbf{d}_{A^*} a_1(\alpha_1, \mathcal{L}_{a_2} \alpha_2). \end{aligned}$$

Proof. First choose $\alpha_1, \alpha_2 \in \Gamma(A^*)$. We have $\mathbf{d}_{D \nmid A}[\alpha_1^\dagger, \alpha_2^\dagger] = 0$. For $\beta_1, \beta_2 \in \Gamma(B^*)$ and $a_1, a_2 \in \Gamma(A)$, we find using Lemma 4.8

$$\begin{aligned} [\mathbf{d}_{D \nmid A} \alpha_1^\dagger, \alpha_2^\dagger](\beta_1^\dagger, \beta_2^\dagger) &= -(\partial_A^* \alpha_2)^\dagger(\mathbf{d}_{D \nmid A} \alpha_1^\dagger(\beta_1^\dagger, \beta_2^\dagger)) + \mathbf{d}_{D \nmid A} \alpha_1^\dagger(\mathcal{L}_{\alpha_2^\dagger} \beta_1^\dagger, \beta_2^\dagger) \\ &\quad + \mathbf{d}_{D \nmid A} \alpha_1^\dagger(\beta_1^\dagger, \mathcal{L}_{\alpha_2^\dagger} \beta_2^\dagger) = 0, \\ [\mathbf{d}_{D \nmid A} \alpha_1^\dagger, \alpha_2^\dagger](\sigma_A^*(a_1), \beta_2^\dagger) &= -(\partial_A^* \alpha_2)^\dagger(\mathbf{d}_{D \nmid A} \alpha_1^\dagger(\sigma_A^*(a_1), \beta_2^\dagger)) + \mathbf{d}_{D \nmid A} \alpha_1^\dagger(\mathcal{L}_{\alpha_2^\dagger} \sigma_A^*(a_1), \beta_2^\dagger) \\ &\quad + \mathbf{d}_{D \nmid A} \alpha_1^\dagger(\sigma_A^*(a_1), \mathcal{L}_{\alpha_2^\dagger} \beta_2^\dagger) = 0 \end{aligned}$$

and

$$\begin{aligned} &[\mathbf{d}_{D \nmid A} \alpha_1^\dagger, \alpha_2^\dagger](\sigma_A^*(a_1), \sigma_A^*(a_2)) \\ &= -(\partial_A^* \alpha_2)^\dagger(\mathbf{d}_{D \nmid A} \alpha_1^\dagger(\sigma_A^*(a_1), \sigma_A^*(a_2))) + \mathbf{d}_{D \nmid A} \alpha_1^\dagger(\mathcal{L}_{\alpha_2^\dagger} \sigma_A^*(a_1), \sigma_A^*(a_2)) \\ &\quad + \mathbf{d}_{D \nmid A} \alpha_1^\dagger(\sigma_A^*(a_1), \mathcal{L}_{\alpha_2^\dagger} \sigma_A^*(a_2)) \\ &= -(\partial_A^* \alpha_2)^\dagger q_{C^*}^* (-\rho_A(a_1)\langle \alpha_1, a_2 \rangle + \rho_A(a_2)\langle \alpha_1, a_1 \rangle + \langle \alpha_1, [a_1, a_2] \rangle) \\ &\quad + \mathbf{d}_{D \nmid A} \alpha_1^\dagger(-\langle a_1, \nabla^* \alpha_2 \rangle^\dagger, \sigma_A^*(a_2)) + \mathbf{d}_{D \nmid A} \alpha_1^\dagger(\sigma_A^*(a_1), -\langle a_2, \nabla^* \alpha_2 \rangle^\dagger) = 0. \end{aligned}$$

Thus, we have $\mathbf{d}_{D \nmid A}[\alpha_1^\dagger, \alpha_2^\dagger] = 0 = [\mathbf{d}_{D \nmid A} \alpha_1^\dagger, \alpha_2^\dagger] + [\alpha_1^\dagger, \mathbf{d}_{D \nmid A} \alpha_2^\dagger]$. Choose now $\alpha \in \Gamma(A^*)$ and $b \in \Gamma(B)$. We have $\mathbf{d}_{D \nmid A}[\sigma_B^*(b), \alpha^\dagger] = \mathbf{d}_{D \nmid A}(\nabla_b^* \alpha)^\dagger$, and so in particular $\mathbf{d}_{D \nmid A}[\sigma_B^*(b), \alpha^\dagger](\beta_1^\dagger, \beta_2^\dagger) = 0$, $\mathbf{d}_{D \nmid A}[\sigma_B^*(b), \alpha^\dagger](\sigma_A^*(a_1), \beta_2^\dagger) = 0$ and

$$\begin{aligned} &\mathbf{d}_{D \nmid A}[\sigma_B^*(b), \alpha^\dagger](\sigma_A^*(a_1), \sigma_A^*(a_2)) \\ &= q_{C^*}^* (-\rho_A(a_1)\langle \nabla_b^* \alpha, a_2 \rangle + \rho_A(a_2)\langle \nabla_b^* \alpha, a_1 \rangle + \langle \nabla_b^* \alpha, [a_1, a_2] \rangle) \\ &= -q_{C^*}^* (\mathbf{d}_A(\nabla_b^* \alpha)(a_1, a_2)). \end{aligned}$$

On the other hand, we can check as above that $[\mathbf{d}_{D\uparrow A}\sigma_B^*(b), \alpha^\dagger](\beta_1^\dagger, \beta_2^\dagger) = 0$, $[\mathbf{d}_{D\uparrow A}\sigma_B^*(b), \alpha^\dagger](\sigma_A^*(a_1), \beta_2^\dagger) = 0$ and

$$\begin{aligned}
& [\mathbf{d}_{D\uparrow A}\sigma_B^*(b), \alpha^\dagger](\sigma_A^*(a_1), \sigma_A^*(a_2)) \\
&= -(\partial_A^*\alpha)^\dagger \left(\mathbf{d}_{D\uparrow A}\sigma_B^*(b)(\sigma_A^*(a_1), \sigma_A^*(a_2)) \right) \\
&\quad + \mathbf{d}_{D\uparrow A}\sigma_B^*(b)(\mathcal{L}_{\alpha^\dagger}\sigma_A^*(a_1), \sigma_A^*(a_2)) + \mathbf{d}_{D\uparrow A}\sigma_B^*(b)(\sigma_A^*(a_1), \mathcal{L}_{\alpha^\dagger}\sigma_A^*(a_2)) \\
&= (\partial_A^*\alpha)^\dagger \langle [\sigma_A^*(a_1), \sigma_A^*(a_2)], \sigma_B^*(b) \rangle \\
&\quad + \mathbf{d}_{D\uparrow A}\sigma_B^*(b)(-\langle a_1, \nabla^*\alpha \rangle^\dagger, \sigma_A^*(a_2)) + \mathbf{d}_{D\uparrow A}\sigma_B^*(b)(\sigma_A^*(a_1), -\langle a_2, \nabla^*\alpha \rangle^\dagger) \\
&= q_{C^*}^* \left(\langle \partial_A^*\alpha, R(a_1, a_2)b \rangle + \rho_A(a_2)\langle a_1, \nabla_b^*\alpha \rangle - \langle b, \nabla_{a_2}^*\langle a_1, \nabla^*\alpha \rangle \rangle \right. \\
&\quad \left. - \rho_A(a_1)\langle a_2, \nabla_b^*\alpha \rangle + \langle b, \nabla_{a_1}^*\langle a_2, \nabla^*\alpha \rangle \rangle \right) \\
&= q_{C^*}^* \left(\langle \partial_A^*\alpha, R(a_1, a_2)b \rangle + \langle a_1, \nabla_{\nabla_{a_2}b}^*\alpha \rangle - \langle a_2, \nabla_{\nabla_{a_1}b}^*\alpha \rangle \right).
\end{aligned}$$

Recall (9) and (10). Using this, we finally get $[\sigma_B^*(b), \mathbf{d}_{D\uparrow A}\alpha^\dagger](\beta_1^\dagger, \beta_2^\dagger) = 0$,

$$\begin{aligned}
& [\sigma_B^*(b), \mathbf{d}_{D\uparrow A}\alpha^\dagger](\sigma_A^*(a_1), \beta_2^\dagger) = \widehat{\nabla_b^*} \left(\mathbf{d}_{D\uparrow A}\alpha^\dagger(\sigma_A^*(a_1), \beta_2^\dagger) \right) \\
&\quad - \mathbf{d}_{D\uparrow A}\alpha^\dagger(\sigma_A^*(\nabla_b a_1) + \widetilde{R(b, \cdot)_{a_1}}, \beta_2^\dagger) - \mathbf{d}_{D\uparrow A}\alpha^\dagger(\sigma_A^*(a_1), \mathcal{L}_b\beta_2^\dagger) = 0
\end{aligned}$$

and

$$\begin{aligned}
& [\sigma_B^*(b), \mathbf{d}_{D\uparrow A}\alpha^\dagger](\sigma_A^*(a_1), \sigma_A^*(a_2)) \\
&= \widehat{\nabla_b^*} \left(\mathbf{d}_{D\uparrow A}\alpha^\dagger(\sigma_A^*(a_1), \sigma_A^*(a_2)) \right) - \mathbf{d}_{D\uparrow A}\alpha^\dagger(\sigma_A^*(\nabla_b a_1) + \widetilde{R(b, \cdot)_{a_1}}, \sigma_A^*(a_2)) \\
&\quad - \mathbf{d}_{D\uparrow A}\alpha^\dagger(\sigma_A^*(a_1), \sigma_A^*(\nabla_b a_2) + \widetilde{R(b, \cdot)_{a_2}}) \\
&= q_{C^*}^* \left(-\rho_B(b)(\mathbf{d}_A\alpha(a_1, a_2)) + \rho_A(\nabla_b a_1)\langle \alpha, a_2 \rangle - \rho_A(a_2)\langle \nabla_b a_1, \alpha \rangle \right. \\
&\quad \left. - \langle \alpha, [\nabla_b a_1, a_2] \rangle + \rho_A(a_1)\langle \nabla_b a_2, \alpha \rangle - \rho_A(\nabla_b a_2)\langle a_1, \alpha \rangle - \langle \alpha, [a_1, \nabla_b a_2] \rangle \right).
\end{aligned}$$

We hence find that

$$\mathbf{d}_{D\uparrow A}[\sigma_B^*(b), \alpha^\dagger] = [\mathbf{d}_{D\uparrow A}\sigma_B^*(b), \alpha^\dagger] + [\sigma_B^*(b), \mathbf{d}_{D\uparrow A}\alpha^\dagger]$$

if and only if

$$\begin{aligned}
& \mathbf{d}_A(\nabla_b^*\alpha)(a_1, a_2) + \langle \partial_A^*\alpha, R(a_1, a_2)b \rangle + \langle a_1, \nabla_{\nabla_{a_2}b}^*\alpha \rangle - \langle a_2, \nabla_{\nabla_{a_1}b}^*\alpha \rangle \\
& - \rho_B(b)(\mathbf{d}_A\alpha(a_1, a_2)) + \rho_A(\nabla_b a_1)\langle \alpha, a_2 \rangle - \rho_A(a_2)\langle \nabla_b a_1, \alpha \rangle - \langle \alpha, [\nabla_b a_1, a_2] \rangle \\
& + \rho_A(a_1)\langle \nabla_b a_2, \alpha \rangle - \rho_A(\nabla_b a_2)\langle a_1, \alpha \rangle - \langle \alpha, [a_1, \nabla_b a_2] \rangle = 0
\end{aligned}$$

for all $a_1, a_2 \in \Gamma(A^*)$. This is

$$\begin{aligned}
& \langle \alpha, \partial_A R(a_1, a_2)b + \nabla_b[a_1, a_2] - [\nabla_b a_1, a_2] - [a_1, \nabla_b a_2] + \nabla_{\nabla_{a_1}b}a_2 - \nabla_{\nabla_{a_2}b}a_1 \rangle \\
& + ([\rho_A(a_1), \rho_B(b)] - \rho_B(\nabla_{a_1}b) + \rho_A(\nabla_b a_1)) \langle \alpha, a_2 \rangle \\
& - ([\rho_A(a_2), \rho_B(b)] - \rho_B(\nabla_{a_2}b) + \rho_A(\nabla_b a_2)) \langle \alpha, a_1 \rangle = 0.
\end{aligned}$$

Hence, using (M5) twice, we obtain (M7) since α was arbitrary.

We conclude the proof of the theorem with the most technical formula. Choose $b_1, b_2 \in \Gamma(B)$. We want to study the equation

$$(18) \quad \mathbf{d}_{D\uparrow A}[\sigma_B^*(b_1), \sigma_B^*(b_2)] = [\mathbf{d}_{D\uparrow A}\sigma_B^*(b_1), \sigma_B^*(b_2)] + [\sigma_B^*(b_1), \mathbf{d}_{D\uparrow A}\sigma_B^*(b_2)].$$

We have $\mathbf{d}_{D \downarrow A} [\sigma_B^*(b_1), \sigma_B^*(b_2)] = \mathbf{d}_{D \downarrow A} (\sigma_B^*[b_1, b_2] + R(b_1, b_2)^{* \dagger})$ and we find easily that both sides of (18) vanish on $\beta_1^\dagger, \beta_2^\dagger$, for $\beta_1, \beta_2 \in \Gamma(B^*)$. We have for $a \in \Gamma(A)$ and $\beta \in \Gamma(B^*)$:

$$\begin{aligned} & \mathbf{d}_{D \downarrow A} (\sigma_B^*[b_1, b_2] + \widetilde{R(b_1, b_2)^*}) (\sigma_A^*(a), \beta^\dagger) \\ &= q_{C^*}^* (\rho_A(a) \langle [b_1, b_2], \beta \rangle + \langle \partial_B^* \beta, R(b_1, b_2)a \rangle - \langle [b_1, b_2], \nabla_a^* \beta \rangle) \\ &= q_{C^*}^* (\langle \nabla_a [b_1, b_2], \beta \rangle + \langle \partial_B^* \beta, R(b_1, b_2)a \rangle) \end{aligned}$$

and

$$\begin{aligned} & [\mathbf{d}_{D \downarrow A} \sigma_B^*(b_1), \sigma_B^*(b_2)] (\sigma_A^*(a), \beta^\dagger) = -\widehat{\nabla_{b_2}^*} (\mathbf{d}_{D \downarrow A} \sigma_B^*(b_1) (\sigma_A^*(a), \beta^\dagger)) \\ & \quad + \mathbf{d}_{D \downarrow A} \sigma_B^*(b_1) (\sigma_A^*(\nabla_{b_2} a) + \widetilde{R(b_2, \cdot)a}, \beta^\dagger) \\ & \quad + \mathbf{d}_{D \downarrow A} \sigma_B^*(b_1) (\sigma_A^*(a), \mathcal{L}_{b_2} \beta^\dagger) \\ & \stackrel{(10)}{=} q_{C^*}^* (-\rho_B(b_2) \rho_A(a) \langle b_1, \beta \rangle + \rho_B(b_2) \langle b_1, \nabla_a^* \beta \rangle + \rho_A(\nabla_{b_2} a) \langle b_1, \beta \rangle - \langle \partial_B^* \beta, \widetilde{R(b_2, b_1)a} \rangle \\ & \quad - \langle b_1, \nabla_{b_2 a}^* \beta \rangle + \langle \partial_B^* \beta, \widetilde{R(b_2, b_1)a} \rangle + \rho_A(a) \langle b_1, \mathcal{L}_{b_2} \beta \rangle - \langle b_1, \nabla_a^* \mathcal{L}_{b_2} \beta \rangle) \\ &= q_{C^*}^* (-\rho_B(b_2) \langle \nabla_a b_1, \beta \rangle + \langle \nabla_{\nabla_{b_2} a} b_1, \beta \rangle + \langle \nabla_a b_1, \mathcal{L}_{b_2} \beta \rangle) \\ &= q_{C^*}^* (-\langle [b_2, \nabla_a b_1], \beta \rangle + \langle \nabla_{\nabla_{b_2} a} b_1, \beta \rangle). \end{aligned}$$

Thus, we find that the two sides of (18) are equal on $(\sigma_A^*(a), \beta^\dagger)$ if and only if (M8) is satisfied.

Finally we consider $a_1, a_2 \in \Gamma(A)$. We have

$$\begin{aligned} & \mathbf{d}_{D \downarrow A} (\sigma_B^*[b_1, b_2] + \widetilde{R(b_1, b_2)^*}) (\sigma_A^*(a_1), \sigma_A^*(a_2)) \\ &= -\widehat{\nabla_{a_1}^*} \ell_{R(b_1, b_2)a_2} + \widehat{\nabla_{a_2}^*} \ell_{R(b_1, b_2)a_1} + \ell_{R(b_1, b_2)[a_1, a_2] - R(a_1, a_2)[b_1, b_2]} = \ell_c \end{aligned}$$

where $c = -\nabla_{a_1}(R(b_1, b_2)a_2) + \nabla_{a_2}(R(b_1, b_2)a_1) + R(b_1, b_2)[a_1, a_2] - R(a_1, a_2)[b_1, b_2] \in \Gamma(C)$, and

$$\begin{aligned} & [\mathbf{d}_{D \downarrow A} \sigma_B^*(b_1), \sigma_B^*(b_2)] (\sigma_A^*(a_1), \sigma_A^*(a_2)) = -\widehat{\nabla_{b_2}^*} (\mathbf{d}_{D \downarrow A} \sigma_B^*(b_1) (\sigma_A^*(a_1), \sigma_A^*(a_2))) \\ & \quad + \mathbf{d}_{D \downarrow A} \sigma_B^*(b_1) (\sigma_A^*(\nabla_{b_2} a_1) + \widetilde{R(b_2, \cdot)a_1}, \sigma_A^*(a_2)) \\ & \quad + \mathbf{d}_{D \downarrow A} \sigma_B^*(b_1) (\sigma_A^*(a_1), \sigma_A^*(\nabla_{b_2} a_2) + \widetilde{R(b_2, \cdot)a_2}) \\ & \stackrel{(9)}{=} \widehat{\nabla_{b_2}^*} \ell_{R_{AB}(a_1, a_2)b_1} - \widehat{\nabla_{a_2}^*} \ell_{R(b_2, b_1)a_1} - \langle \widetilde{R(\nabla_{b_2} a_1, a_2)^*} - \nabla_{a_2}^{\text{Hom}}(\widetilde{R(b_2, \cdot)a_1}), \sigma_B^*(b_1) \rangle \\ & \quad + \widehat{\nabla_{a_1}^*} \ell_{R(b_2, b_1)a_2} + \langle \widetilde{R(\nabla_{b_2} a_2, a_1)^*} - \nabla_{a_1}^{\text{Hom}}(\widetilde{R(b_2, \cdot)a_2}), \sigma_B^*(b_1) \rangle \\ &= \ell_{c_1 + c_2}, \end{aligned}$$

where

$$\begin{aligned} c_1 &= \nabla_{b_2}(R(a_1, a_2)b_1) - \nabla_{a_2}(R(b_2, b_1)a_1) - R(\nabla_{b_2} a_1, a_2)b_1 + \langle b_1, \nabla_{a_2}^{\text{Hom}}(R(b_2, \cdot)a_1) \rangle \\ &= \nabla_{b_2}(R(a_1, a_2)b_1) + R(\nabla_{a_2} b_1, b_2)a_1 - R(\nabla_{b_2} a_1, a_2)b_1, \\ c_2 &= \nabla_{a_1}(R(b_2, b_1)a_2) + R(\nabla_{b_2} a_2, a_1)b_1 - \langle b_1, \nabla_{a_1}^{\text{Hom}}(R(b_2, \cdot)a_2) \rangle \\ &= R(\nabla_{a_1} b_1, b_2)a_2 - R(a_1, \nabla_{b_2} a_2)b_1. \end{aligned}$$

Hence, we find that the two sides of (18) coincides on $(\sigma_A^*(a_1), \sigma_A^*(a_2))$ if and only if (M9) is satisfied. \square

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